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## Hyperelliptic curves with prescribed $p$ -torsion

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**Abstract.** In this paper, we show that there exist families of curves (defined over an algebraically closed field  $k$  of characteristic  $p > 2$ ) whose Jacobians have interesting  $p$ -torsion. For example, for every  $0 \leq f \leq g$ , we find the dimension of the locus of hyperelliptic curves of genus  $g$  with  $p$ -rank at most  $f$ . We also produce families of curves so that the  $p$ -torsion of the Jacobian of each fibre contains multiple copies of the group scheme  $\alpha_p$ . The method is to study curves which admit an action by  $(\mathbb{Z}/2)^n$  so that the quotient is a projective line. As a result, some of these families intersect the hyperelliptic locus  $\mathcal{H}_g$ .

### 1. Introduction

When investigating abelian varieties defined over an algebraically closed field  $k$  of characteristic  $p$ , it is natural to study the invariants related to their  $p$ -torsion such as their  $p$ -rank or  $a$ -number. Such invariants are well-understood and have been used to define stratifications of the moduli space  $\mathcal{A}_g$  of principally polarized abelian varieties of dimension  $g$ . There is a deep interest in understanding whether the Torelli locus intersects such strata in  $\mathcal{A}_g$ . More generally, one can ask for the dimension of the intersection of these strata with the image of the moduli spaces  $\mathcal{M}_g$  or  $\mathcal{H}_g$  under the Torelli map. In this paper, we show that the Torelli locus intersects several of these strata by producing families of curves so that the  $p$ -torsion of the Jacobian of each fibre contains certain group schemes.

Recall that the group scheme  $\mu_p = \mu_{p,k}$  is the kernel of Frobenius on  $\mathbb{G}_m$  and the group scheme  $\alpha_p = \alpha_{p,k}$  is the kernel of Frobenius on  $\mathbb{G}_a$ . As schemes,  $\mu_p \simeq \text{Spec}(k[x]/(x-1)^p)$  and  $\alpha_p \simeq \text{Spec}(k[x]/x^p)$  over  $k$ . If  $\text{Jac}(X)$  is the Jacobian of a  $k$ -curve  $X$ , the  $p$ -rank of  $X$  is  $\dim_{\mathbb{F}_p} \text{Hom}(\mu_p, \text{Jac}(X))$  and the  $a$ -number of  $X$  is  $\dim_k \text{Hom}(\alpha_p, \text{Jac}(X))$ .

Let  $V_{g,f}$  denote the sublocus of  $\overline{\mathcal{M}}_g$  consisting of curves of genus  $g$  with  $p$ -rank at most  $f$ . For every  $g$  and every  $0 \leq f \leq g$ , the locus  $V_{g,f}$  has codimension  $g - f$  in  $\overline{\mathcal{M}}_g$ , [1]. In Section 2, we use results from [1] to prove that there exist smooth hyperelliptic curves of genus  $g$  with every possible  $p$ -rank  $f$ .

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**Theorem 1.** *For all  $g \geq 1$  and all  $0 \leq f \leq g$ , the locus  $V_{g,f} \cap \mathcal{H}_g$  is non-empty of dimension  $g - 1 + f$ . In particular, there exists a smooth hyperelliptic curve of genus  $g$  and  $p$ -rank  $f$ .*

Let  $T_{g,a}$  denote the sublocus of  $\overline{\mathcal{M}}_g$  consisting of curves of genus  $g$  with  $a$ -number at least  $a$ . In Section 5, we show that  $T_{g,a}$  is non-empty under certain conditions on  $g$  and  $a$  by producing curves  $X$  so that  $\text{Jac}(X)[p]$  contains multiple copies of  $\alpha_p$ . Let  $\mathcal{H}_{g,n}$  be the sublocus of the moduli space  $\mathcal{M}_g$  consisting of smooth curves of genus  $g$  which admit an action by  $(\mathbb{Z}/2)^n$  so that the quotient is a projective line.

**Corollary 3.** *Suppose  $n \geq 2$  and  $p \geq 2n + 1$ . Suppose  $g$  is such that  $\mathcal{H}_{g,n}$  is non-empty of dimension at least  $n + 1$ . Then the intersection  $\mathcal{H}_{g,n} \cap T_{g,n}$  has codimension at most  $n$  in  $\mathcal{H}_{g,n}$ . In particular, there exists a smooth curve of genus  $g$  with  $a$ -number at least  $n$ .*

The dimension of the family in Corollary 3 is at least  $(g + 2^n - 1)/2^{n-2} - 3 - n$ . The precise numerical conditions for  $g$  can be found in Section 5. The main interest in this result is not only that certain group schemes occur in the  $p$ -torsion of the Jacobians, but also that the dimension of the families is large in comparison with the dimension of  $\mathcal{H}_{g,n}$ .

For small values of  $n$ , we further show that these families of curves intersect the hyperelliptic locus  $\mathcal{H}_g$ , resulting in the following corollaries.

**Corollary 4.** *Suppose  $g \geq 2$  and  $p \geq 5$ . There exists a  $(g - 2)$ -dimensional family of smooth hyperelliptic curves of genus  $g$  whose fibres have  $a$ -number 2 and  $p$ -rank  $g - 2$ .*

**Corollary 5.** *Suppose  $g \geq 5$  is odd and  $p \geq 7$ . There exists a  $(g - 5)/2$ -dimensional family of smooth hyperelliptic curves of genus  $g$  whose fibres have  $a$ -number at least 3.*

In Section 6, we consider the problem of constructing Jacobians whose  $p$ -torsion contains group schemes other than  $\mu_p$  or  $\alpha_p$ . We prove that for all  $g \geq 2$  there exists a smooth hyperelliptic curve of genus  $g$  whose  $p$ -torsion contains the group scheme corresponding to a supersingular non-superspecial abelian surface. We describe this group scheme and its covariant Dieudonné module in Section 6. It has  $a$ -number 1 and  $p$ -rank 0.

Our method for these results is to analyze the curves in the locus  $\mathcal{H}_{g,n}$  in terms of fibre products of hyperelliptic curves. In Section 3, we extend results of Kani and Rosen [8] to compare the  $p$ -torsion of the Jacobian of a curve  $X$  in  $\mathcal{H}_{g,n}$  to the  $p$ -torsion of the Jacobians of its  $\mathbb{Z}/2\mathbb{Z}$ -quotients up to isomorphism. We then use Yui's description of the branch locus of a non-ordinary hyperelliptic curve, [17]. In some cases, this reduces the study of the  $p$ -torsion of the Jacobian of  $X$  to the study of the intersection of some subvarieties in the configuration space of branch points. We consider this in Section 4.

Throughout,  $k$  is an algebraically closed field of characteristic  $p > 2$ . We assume  $g \geq 1$  to avoid trivial cases. Without further comment, we will speak of a fibre of a relative curve when we mean a geometric fibre.

This paper led us to pose some open questions on this topic in [2].

## 2. Curves with prescribed $p$ -rank

We begin by considering the  $p$ -rank of Jacobians of hyperelliptic curves. Recall that the  $p$ -rank,  $\dim_{\mathbb{F}_p} \text{Hom}(\mu_p, \text{Jac}(X))$ , of a  $k$ -curve  $X$  is an integer between 0 and its genus  $g$ . The curve  $X$  is said to be *ordinary* if it has  $p$ -rank equal to  $g$ . In other words,  $X$  is ordinary if  $\text{Jac}(X)[p] \cong (\mathbb{Z}/p \oplus \mu_p)^g$ . Let  $V_{g,f}$  denote the sublocus of  $\overline{\mathcal{M}}_g$  consisting of curves of genus  $g$  with  $p$ -rank at most  $f$ .

Consider the moduli space  $\mathcal{H}_g$  of smooth hyperelliptic curves of genus  $g$  and its closure  $\overline{\mathcal{H}}_g$  in  $\overline{\mathcal{M}}_g$ . It is known that  $\mathcal{H}_g$  is affine of dimension  $2g - 1$ . Both  $\mathcal{H}_g$  and  $\overline{\mathcal{H}}_g$  are smooth algebraic stacks over  $\mathbb{Z}[1/2]$  (for example, see [16, Proposition 1]). Since  $k$  is an algebraically closed field, this fact implies that if two subvarieties of  $\overline{\mathcal{H}}_g$  intersect then the codimension of their intersection is at most the sum of their codimensions.

The boundary  $\overline{\mathcal{H}}_g - \mathcal{H}_g$  consists of components  $\Delta_0$  and  $\Delta_i$  for integers  $1 \leq i \leq g/2$ . The generic point of  $\Delta_i$  corresponds to the isomorphism class of a singular curve with two irreducible components  $X_i$  and  $X_{g-i}$  intersecting in a node which we denote  $P_i$ . Here  $X_i$  (resp.  $X_{g-i}$ ) is a hyperelliptic curve of genus  $i$  (resp.  $g - i$ ) and the point  $P_i$  is fixed by the hyperelliptic involution on  $X_i$  (resp.  $X_{g-i}$ ). The generic point of  $\Delta_0$  corresponds to the isomorphism class of an irreducible hyperelliptic curve  $X'_0$  with a node. The normalization  $X_0$  of  $X'_0$  is a hyperelliptic curve of genus  $g - 1$ , and the inverse image of the node in  $X'_0$  consists of two distinct points in  $X_0$  which are exchanged by the hyperelliptic involution. Note that  $\text{Jac}(X'_0)$  is a semi-abelian variety and the toric part of its  $p$ -torsion contains a copy of the group scheme  $\mu_p$ . So  $\Delta_0 \cap V_{g,0}$  is empty in  $\overline{\mathcal{M}}_g$ .

We first show that each component of  $V_{g,0} \cap \overline{\mathcal{H}}_g$  has dimension  $g - 1$ .

**Proposition 1.** *The locus  $V_{g,0} \cap \overline{\mathcal{H}}_g$  is pure of codimension  $g$  in  $\overline{\mathcal{H}}_g$ .*

*Proof.* We work by induction on  $g$ . The statement is true in the case  $g = 1$  since the locus of supersingular elliptic curves has dimension 0. Assume that the statement is true for all  $g' < g$ , and consider any component  $C_0$  of the intersection  $V_{g,0} \cap \overline{\mathcal{H}}_g$ . By the purity argument of [10, 1.6], the codimension of  $C_0$  in  $\overline{\mathcal{H}}_g$  is at most  $g$ . Furthermore,  $C_0$  intersects the boundary of  $\overline{\mathcal{H}}_g$  because  $\mathcal{H}_g$  is affine. Since  $C_0$  does not intersect  $\Delta_0$ , it must intersect  $\Delta_i$  for some  $1 \leq i \leq g/2$ . We fix one such  $\Delta_i$  and consider the dimension of the intersection.

A curve corresponding to a point in the intersection of  $C_0$  and  $\Delta_i$  is formed from two hyperelliptic curves  $X_i$  and  $X_{g-i}$  which must both have  $p$ -rank 0. Thus  $X_i$  corresponds to a point of  $V_{i,0} \cap \overline{\mathcal{H}}_i$  and likewise  $X_{g-i}$  corresponds to a point of  $V_{g-i,0} \cap \overline{\mathcal{H}}_{g-i}$ . By the inductive hypothesis, there is at most an  $i - 1$  (resp.  $g - i - 1$ ) dimensional family of choices for  $X_i$  (resp.  $X_{g-i}$ ). Since  $X_i$  and  $X_{g-i}$  intersect in a unique point  $P_i$ , this point must be fixed under the hyperelliptic involutions of the two curves. Thus there are only finitely many choices for the point  $P_i$ . It follows that  $\dim(C_0 \cap \Delta_i) \leq (i - 1) + (g - i - 1) + 0 = g - 2$  and the codimension of  $C_0 \cap \Delta_i$  in  $\overline{\mathcal{H}}_g$  is at least  $g + 1$ .

We can deduce that  $\text{codim}(C_0 \cap \Delta_i) \leq \text{codim}(C_0) + 1$  in  $\overline{\mathcal{H}}_g$  from the fact that  $\Delta_i$  has codimension 1 in  $\overline{\mathcal{H}}_g$ . This implies that the codimension of  $C_0$  in  $\overline{\mathcal{H}}_g$  is exactly  $g$  and therefore that  $V_{g,0} \cap \overline{\mathcal{H}}_g$  is pure of codimension  $g$  in  $\overline{\mathcal{H}}_g$ .  $\square$

Next we show that each component of  $V_{g,f} \cap \overline{\mathcal{H}}_g$  has dimension  $g - 1 + f$  (for  $g \geq 1$ ).

**Proposition 2.** *The locus  $V_{g,f} \cap \overline{\mathcal{H}}_g$  is pure of codimension  $g - f$  in  $\overline{\mathcal{H}}_g$ .*

*Proof.* By Proposition 1, we can suppose  $f \geq 1$ . Consider a component  $C_0$  of  $V_{g,f} \cap \overline{\mathcal{H}}_g$ . By [10, 1.6],  $C_0$  has codimension at most  $g - f$  in  $\overline{\mathcal{H}}_g$  and thus dimension at least  $g - 1 + f$ . Because  $p > 2$ , a complete subvariety of  $\overline{\mathcal{H}}_g - \Delta_0$  has dimension at most  $g - 1$ , by [1, Lemma 2.6]. So  $C_0$  intersects  $\Delta_0$ .

A point of  $C_0 \cap \Delta_0$  corresponds to a curve  $X'_0$  self-intersecting in a node  $P_0$ . The normalization  $X_0$  of  $X'_0$  is a hyperelliptic curve of genus  $g - 1$ . Since the toric part of  $\text{Jac}(X'_0)[p]$  contains a copy of the group scheme  $\mu_p$ , this implies that the  $p$ -rank of  $X_0$  is at most  $f - 1$ . So  $X_0$  corresponds to a point of  $V_{g-1,f-1} \cap \overline{\mathcal{H}}_g$ . The choice of the nodal point  $P_0$  is equivalent to a choice of two distinct points of  $X_0$  which are exchanged by the hyperelliptic involution. So  $\dim(C_0 \cap \Delta_0) = \dim(V_{g-1,f-1} \cap \overline{\mathcal{H}}_g) + 1$ .

Furthermore,  $\text{codim}(C_0) \geq \text{codim}(C_0 \cap \Delta_0) - \text{codim}(\Delta_0)$  in  $\overline{\mathcal{H}}_g$ . A calculation shows that the codimension of  $C_0$  in  $\overline{\mathcal{H}}_g$  is at least the codimension of  $V_{g-1,f-1} \cap \overline{\mathcal{H}}_{g-1}$  in  $\overline{\mathcal{H}}_{g-1}$ . Repeating this calculation, we see that the codimension of  $C_0$  in  $\overline{\mathcal{H}}_g$  is at least the codimension of  $V_{g-f,0} \cap \overline{\mathcal{H}}_{g-f}$  in  $\overline{\mathcal{H}}_{g-f}$ , which by Proposition 1 is  $g - f$ . It follows that  $V_{g,f} \cap \overline{\mathcal{H}}_g$  is pure of codimension  $g - f$  in  $\overline{\mathcal{H}}_g$ .  $\square$

The next Theorem is the main result in the paper on the  $p$ -rank of hyperelliptic curves.

**Theorem 1.** *For all  $g \geq 1$  and all  $0 \leq f \leq g$ , the locus  $V_{g,f} \cap \overline{\mathcal{H}}_g$  is non-empty of dimension  $g - 1 + f$ . In particular, there exists a smooth hyperelliptic curve of genus  $g$  and  $p$ -rank  $f$ .*

*Proof.* By [1, Proposition 2.7], there exists a smooth hyperelliptic curve  $X$  of genus  $g$  and  $p$ -rank equal to zero for all  $g \geq 1$ . For  $0 \leq f \leq g$ , let  $C_f$  be the component of  $V_{g,f} \cap \overline{\mathcal{H}}_g$  containing  $X$ . By Proposition 2,  $C_f$  has codimension  $g - f$  in  $\overline{\mathcal{H}}_g$ . It follows that  $C_f \cap \overline{\mathcal{H}}_g$  has dimension  $g - 1 + f$  since  $C_f$  is not contained in the boundary of  $\overline{\mathcal{H}}_g$ . Now  $C_{f-1}$  has codimension only  $g - f + 1$  in  $\overline{\mathcal{H}}_g$ . So the generic point of  $C_f$  is a smooth hyperelliptic curve with  $p$ -rank exactly  $f$ .  $\square$

We now turn to the question of constructing Jacobians of curves with large  $a$ -number. To do this, we first analyze the Jacobians of fibre products of hyperelliptic curves in Section 3 and then analyze the geometry of the branch points of non-ordinary hyperelliptic curves in Section 4. Unless specified otherwise, the results in the next two sections are also valid in characteristic 0 (but not in characteristic 2).

### 3. Fibre products of hyperelliptic curves

Let  $G$  be an elementary abelian 2-group of order  $2^n$ . In this section, we describe  $G$ -Galois covers  $\phi : X \rightarrow \mathbb{P}_k^1$  where  $X$  is a smooth projective  $k$ -curve of genus  $g$ . For such a cover  $\phi$ , we show that the Jacobian of  $X$  decomposes into  $2^n - 1$  factors which are Jacobians as well. We study some geometric properties of the Hurwitz space  $H_{g,n}$  which parametrizes isomorphism classes of such covers  $\phi$ .

#### 3.1. The moduli space $H_{g,n}$

We first recall a result about the coarse moduli space parametrizing isomorphism classes of  $G$ -Galois covers  $\phi : X \rightarrow \mathbb{P}_k^1$  where  $X$  is a smooth projective  $k$ -curve of genus  $g$ . This description is related to the theory of Hurwitz schemes and gives a framework to describe these covers. In particular, this framework allows one to consider families of such covers with varying branch locus, to lift such a cover from characteristic  $p$  to characteristic 0, or to study the locus in  $\mathcal{M}_g$  of curves with a certain type of action by  $G$ .

To be precise, let  $F_{g,n}$  be the contravariant functor which associates to any  $k$ -scheme  $\Omega$  the set of isomorphism classes of  $(\mathbb{Z}/2)^n$ -Galois covers  $\phi_\Omega : X_\Omega \rightarrow \mathbb{P}_\Omega^1$  where  $X_\Omega$  is a flat  $\Omega$ -curve whose fibres are smooth projective curves of genus  $g$  and where the branch locus  $B$  of  $\phi_\Omega$  is a simple horizontal divisor. In other words, the branch locus consists of  $\Omega$ -points of  $\mathbb{P}_\Omega^1$  which do not intersect. Since each inertia group is a cyclic group of order 2, the Riemann-Hurwitz formula implies  $g = 2^{n-2}|B| - 2^n + 1$ . The following facts about the Hurwitz scheme which coarsely represents this functor are well-known over the complex numbers.

- Lemma 1.** i) *There exists a coarse moduli space  $H_{g,n}$  for the functor  $F_{g,n}$  which is of finite type over  $\mathbb{Z}[1/2]$ .*  
 ii) *There is a natural morphism  $\tau : H_{g,n} \rightarrow \mathcal{M}_g$  whose fibres have dimension three.*  
 iii) *There is a natural morphism  $\beta : H_{g,n} \rightarrow \mathbb{P}^{|B|}$  which is proper and étale over the image.*

*Proof.* See [15, Chapter 10] for the construction of  $H_{g,n}$  and the morphisms  $\tau$  and  $\beta$  over  $\mathbb{C}$ . The corresponding statements over  $\mathbb{Z}[1/2]$  follow from [16, Theorem 4].

We recall some of the details about the morphisms  $\tau$  and  $\beta$ . The morphism  $\tau$  associates to any  $\Omega$ -point of  $H_{g,n}$  the isomorphism class of  $X_\Omega$ , where  $\phi_\Omega : X_\Omega \rightarrow \mathbb{P}_\Omega^1$  is the corresponding cover of  $\Omega$ -curves. The fibres have dimension three since  $X_\Omega$  is isotrivial if and only if after an étale base change from  $\Omega$  to  $\Omega'$  there is a projective linear transformation  $\rho$  such that  $\rho\phi_{\Omega'}$  is constant, [12, Lemma 2.1.2].

The morphism  $\beta$  associates to any  $\Omega$ -point of  $H_{g,n}$  the  $\Omega$ -point of the configuration space  $\mathbb{P}^{|B|}$  determined by the branch locus of the associated cover. More specifically,  $\beta$  associates to any cover  $\phi_\Omega : X_\Omega \rightarrow \mathbb{P}_\Omega^1$  the  $\Omega$ -point  $[a_0 : \dots : a_{|B|}]$  of  $\mathbb{P}^{|B|}$  where  $a_i$  are the coefficients of the polynomial whose roots are the branch points of  $\phi_\Omega$ . Note that the  $k$ -points of the image of  $\beta$  correspond to polynomials with no multiple roots.  $\square$

We denote by  $\mathcal{H}_{g,n}$  the image  $\tau(H_{g,n})$  in  $\mathcal{M}_g$ . Given a smooth connected  $k$ -curve  $X$ , then  $X$  corresponds to a point of  $\mathcal{H}_{g,n}$  if and only if there exists a subgroup  $G \subset \text{Aut}(X)$  with quotient  $X/G \simeq \mathbb{P}^1$ . Note that  $\mathcal{H}_{g,1}$  is simply the locus  $\mathcal{H}_g$  of hyperelliptic curves in  $\mathcal{M}_g$ .

It is often more useful to describe the branch locus of  $\phi_\Omega$  directly as an  $\Omega$ -point of  $(\mathbb{P}^1)^{|B|}$ . This can be done by considering an ordering of the branch points of  $\phi_\Omega$ . The branch locus of a cover corresponding to a  $k$ -point of  $H_{g,n}$  can be any  $k$ -point of  $(\mathbb{P}^1)^{|B|} - \Delta$  where  $\Delta$  is the weak diagonal consisting of points having at least two equal coordinates. In particular, for any  $\Omega$ -point  $(b_1, \dots, b_{2g+2})$  of  $(\mathbb{P}^1)^{2g+2} - \Delta$  there is a unique hyperelliptic cover  $\phi_\Omega : X_\Omega \rightarrow \mathbb{P}^1_\Omega$  branched at  $\{b_1, \dots, b_{2g+2}\}$ . Also the curve  $X_\Omega$  has genus  $g$ .

### 3.2. The fibres of $H_{g,n}$

We now describe some properties of a  $G$ -Galois cover  $\phi : X \rightarrow \mathbb{P}^1$  corresponding to a point of  $H_{g,n}$ . In fact, the cover  $\phi$  arises as the fibre product of  $n$  hyperelliptic covers which satisfy a strong disjointness condition on their branch loci.

Consider an isomorphism  $\iota : (\mathbb{Z}/2)^n \simeq G$ . For  $i \in \{1, \dots, n\}$ , this isomorphism determines a natural element  $s_i$  of order 2 in  $G$ . Let  $H_i \simeq (\mathbb{Z}/2)^{n-1}$  be the subgroup generated by all  $s_j$  for  $j \neq i$ . Suppose for  $i \in \{1, \dots, n\}$  that  $B_i$  is a non-empty finite subset of  $\mathbb{P}^1$  of even cardinality. For any non-empty  $S \subset \{1, \dots, n\}$ , denote by  $B_S$  the set of all  $b \in \mathbb{P}^1$  such that  $b \in B_i$  for an odd number of  $i \in S$  and denote by  $C_S \rightarrow \mathbb{P}^1$  the hyperelliptic cover branched at  $B_S$ . Finally, let  $H_S$  be the subgroup of  $G$  consisting of all elements  $\sum_{i=1}^n a_i s_i$  such that  $\sum_{i \in S} a_i$  is even. Note that each  $H_S$  is non-canonically isomorphic to  $(\mathbb{Z}/2)^{n-1}$ . Furthermore, when  $S = \{i\}$  we have  $B_S = B_i$  and  $H_S = H_i$ .

**Lemma 2.** *Suppose  $\phi : X \rightarrow \mathbb{P}^1$  is the normalized fibre product over  $\mathbb{P}^1$  of  $n$  smooth hyperelliptic covers  $C_i \rightarrow \mathbb{P}^1$  with branch loci  $B_i$ . Then  $\phi$  is a  $G$ -Galois cover and the quotient of  $X$  by  $H_S$  is the hyperelliptic cover  $C_S \rightarrow \mathbb{P}^1$  branched at  $B_S$ .*

*Proof.* The cover  $\phi : X \rightarrow \mathbb{P}^1$  is a  $G$ -Galois cover of (possibly disconnected) smooth curves by the definition of the fibre product. Also by definition,  $C_i \rightarrow \mathbb{P}^1$  is the quotient of  $X$  by the subgroup  $H_i$ .

The branch locus  $B$  of  $\phi$  equals  $\cup_{i=1}^n B_i$ . For  $b \in B$ , the inertia group  $I_b$  of  $X \rightarrow \mathbb{P}^1$  above  $b$  must be cyclic; thus  $I_b \simeq \mathbb{Z}/2$ . In fact, the generator  $(\alpha_1, \dots, \alpha_n)$  of  $I_b$  satisfies  $\alpha_i = 1$  if and only if  $b \in B_i$ . To see this, note that if  $b \in B_i$ , then  $C_i \rightarrow \mathbb{P}^1$  is branched at  $b$  and so  $I_b \not\subset H_i$ ; it follows that  $\alpha_i = 1$  if  $b \in B_i$ . On the other hand, if  $b \notin B_i$ , then  $C_i \rightarrow \mathbb{P}^1$  is unramified at  $b$  and so  $I_b \subset H_i$ ; it follows that  $\alpha_i = 0$  if  $b \notin B_i$ .

Since  $H_S \simeq (\mathbb{Z}/2)^{n-1}$ , the quotient  $X/H_S \rightarrow \mathbb{P}^1$  is hyperelliptic; it remains to show that the branch locus of this cover is  $B_S$ . For  $b \in B$ , the cover  $X/H_S \rightarrow \mathbb{P}^1$  is branched at  $b$  if and only if  $I_b \not\subset H_S$ , which is equivalent to  $(\alpha_1, \dots, \alpha_n) \notin H_S$ . So  $X/H_S \rightarrow \mathbb{P}^1$  is branched at  $b$  if and only if  $\sum_{i \in S} \alpha_i$  is odd. Now,  $\sum_{i \in S} \alpha_i \equiv \#\{i \in S | b \in B_i\} \pmod{2}$ , so this number is odd if and only if  $b \in B_i$  for an odd number of  $i \in S$ . Thus  $X/H_S$  is branched at  $B_S$  by definition.  $\square$

In Section 5, we construct covers  $\phi : X \rightarrow \mathbb{P}^1$  corresponding to points of  $H_{g,n}$  for which  $X$  is also hyperelliptic. For example, when  $n = 2$ , suppose  $\phi$  is the normalized fibre product of two hyperelliptic covers  $\phi_1$  and  $\phi_2$ . The curve  $X$  will also be hyperelliptic if its quotient  $C_{1,2} = X/H_{1,2}$  is isomorphic to  $\mathbb{P}^1$ . This occurs when  $g_1 = g_2$  and  $B_1$  and  $B_2$  overlap in all but one point; or when  $g_2 = g_1 + 1$  and  $B_1 \subset B_2$ . The other extreme is considered in [14] where Stepanov uses the fibre product of two hyperelliptic curves whose branch loci intersect in a single point to construct Goppa codes.

We say that the collection  $\{B_i\}_{i=1}^n$  is *strongly disjoint* if the following two conditions are satisfied: first, the sets  $B_S$  are distinct for all non-empty  $S \subset \{1, \dots, n\}$ ; second,  $B = \cup_{i=1}^n B_i$  is a simple horizontal divisor. In other words, if  $b_1, b_2 \in B$  are two  $\Omega$ -points of  $\mathbb{P}^1_\Omega$  for some scheme  $\Omega$ , then the second condition insures that either  $b_1 = b_2$  or that  $b_1$  and  $b_2$  do not intersect in  $\mathbb{P}^1_\Omega$ .

**Lemma 3.** *A cover  $\phi : X \rightarrow \mathbb{P}^1$  corresponds to a point of  $H_{g,n}$  if and only if  $X$  has genus  $g$  and  $\phi : X \rightarrow \mathbb{P}^1$  is isomorphic to the normalized fibre product over  $\mathbb{P}^1$  of  $n$  smooth hyperelliptic covers  $C_i \rightarrow \mathbb{P}^1$  whose branch loci  $B_i$  form a strongly disjoint set.*

*Proof.* If  $\phi : X \rightarrow \mathbb{P}^1$  is the normalized fibre product of  $n$  hyperelliptic covers with branch loci  $B_i$ , then it is clear that  $\phi$  is a  $G$ -Galois cover and  $X$  is projective. Furthermore,  $C_j \rightarrow \mathbb{P}^1$  is disjoint from the normalized fibre product of all  $C_i \rightarrow \mathbb{P}^1$  for  $i \leq j$ ; otherwise, by Lemma 2,  $C_j \rightarrow \mathbb{P}^1$  would be isomorphic to  $C_S \rightarrow \mathbb{P}^1$  for some  $\{j\} \neq S \subset \{1, \dots, n\}$ . This would imply  $B_S = B_j$  for some  $S \neq \{j\}$  which would contradict the fact that  $\{B_i\}$  form a strongly disjoint set. Since these covers are disjoint over  $\mathbb{P}^1$ , it follows that  $X$  is connected. Also  $X$  is a smooth relative curve since  $B = \cup_{i=1}^n B_i$  is a simple horizontal divisor. By hypothesis,  $X$  has genus  $g$  and so  $\phi$  corresponds to a point of  $H_{g,n}$ .

Conversely, if  $\phi : X \rightarrow \mathbb{P}^1$  corresponds to a point of  $H_{g,n}$ , then  $X$  has genus  $g$  by definition. Consider the quotients  $C_i \rightarrow \mathbb{P}^1$  of  $\phi$  by the subgroups  $H_i$  of  $G$  for  $i = 1, \dots, n$ . These covers are clearly smooth and hyperelliptic. By the universal property of fibre products, there is a morphism from  $X$  to the normalized fibre product of the covers  $C_i \rightarrow \mathbb{P}^1$ . This morphism must be an isomorphism since both  $X$  and the normalized fibre product have degree  $2^n$  over  $\mathbb{P}^1$ . Also,  $\phi : X \rightarrow \mathbb{P}^1$  dominates the fibre product of any two of the quotients  $C_S \rightarrow \mathbb{P}^1$  with branch locus  $B_S$ , by Lemma 2. Since  $X$  is connected, these quotients  $C_S \rightarrow \mathbb{P}^1$  must all be disjoint; in other words, the sets  $B_S$  must all be distinct. Also,  $\cup_{i=1}^n B_i$  is the branch locus  $B$  of  $\phi$ ; by definition,  $B$  is a simple horizontal divisor. Thus  $\{B_i\}$  form a strongly disjoint set.  $\square$

**Corollary 1.** *For  $n \geq 2$ , the locus  $\mathcal{H}_{g,n}$  has dimension  $(g + 2^n - 1)/2^{n-2} - 3$  if  $g \equiv 1 \pmod{2^{n-2}}$  and is empty otherwise. In particular, the dimension of the locus  $\mathcal{H}_{g,2}$  is  $g$ .*

*Proof.* The dimension of  $H_{g,n}$  is equal to the dimension of  $(\mathbb{P}^1)^{|B|}$ , namely the number of branch points  $|B|$  of the corresponding covers. By the Riemann-Hurwitz formula,  $|B| = (g + 2^n - 1)/2^{n-2}$ . By Lemma 1, the dimension of  $\mathcal{H}_{g,n}$  is three less than the dimension of  $H_{g,n}$ , which simplifies to  $g$  when  $n = 2$ .  $\square$

### 3.3. Decomposition of the Jacobian

We will now describe the isogeny class of the Jacobian for any curve  $X$  for which there exists a cover  $\phi : X \rightarrow \mathbb{P}_k^1$  corresponding to a  $k$ -point of  $H_{g,n}$ . For  $i \in \{1, \dots, n\}$ , suppose  $\phi_i : C_i \rightarrow \mathbb{P}_k^1$  is a smooth hyperelliptic cover with branch locus  $B_i$ . Suppose  $\{B_i\}_{i=1}^n$  form a strongly disjoint set and let  $B = \cup_{i=1}^n B_i$ .

**Proposition 3.** *Suppose  $\phi : X \rightarrow \mathbb{P}_k^1$  is the normalization of the fibre product of  $\phi_i$  for  $i = 1, \dots, n$ . Then  $\text{Jac}(X)$  is isogenous to  $\prod(\text{Jac}(C_S))$  where the product is taken over all non-empty  $S \subset \{1, \dots, n\}$ .*

*Proof.* Note that  $X/H_S$  is the hyperelliptic curve  $C_S$  by Lemma 2. Thus the result follows directly from [8, Theorem C] if  $\text{genus}(X) = \sum_S \text{genus}(C_S)$ . By the Riemann-Hurwitz formula,  $\text{genus}(C_S) = -1 + |B_S|/2$ . Since  $B = \cup_{i=1}^n B_i$  is the branch locus of  $X \rightarrow \mathbb{P}_k^1$ , it follows that  $\text{genus}(X) = 2^{n-2}|B| - 2^n + 1$ . The proof follows by showing that  $\sum_S |B_S| = 2^{n-1}|B|$  by the inclusion-exclusion principle.  $\square$

The isogeny between  $\text{Jac}(X)$  and  $\prod(\text{Jac}(C_S))$  is not sufficient to study the  $a$ -number of  $X$  since the  $a$ -number is not an isogeny invariant. For this reason, we now generalize Proposition 3 by showing that the de Rham cohomology group  $H_{\text{dR}}^1(X)$  also decomposes. Equivalently, one could work with the crystalline cohomology group  $H_{\text{crys}}^1(X)$  evaluated at  $k$ , [6, 1.3.6]. We thank Kani [7] for helping us with the proof of Proposition 4. Let  $N = 2^n - 1$ .

**Proposition 4.** *Suppose  $\text{char}(k) \neq 2$ . Then  $H_{\text{dR}}^1(X)$  is isomorphic to  $\oplus_S H_{\text{dR}}^1(C_S)$  as  $k[G]$ -modules, where the sum is taken over all non-empty  $S \subset \{1, \dots, n\}$ .*

*Proof.* Since  $\text{char}(k) \neq 2$ , there exists an idempotent  $\epsilon_S$  corresponding to the subgroup  $H_S$  in the group ring  $k[G]$  for every nonempty subset  $S \subset \{1, \dots, n\}$ . Namely,  $\epsilon_S = \sum h/2^{n-1}$ , where the sum ranges over all  $h \in H_S$ . Let  $\epsilon_G$  be the idempotent  $\sum h/2^n$ , where the sum ranges over all  $h \in G$ . By Lemma 2,  $C_S$  is the quotient of  $X$  by  $H_S$ , so  $H_{\text{dR}}^1(C_S) \cong (H_{\text{dR}}^1(X))^{H_S} \cong \epsilon_S H_{\text{dR}}^1(X)$ . Furthermore, note that  $0 = H_{\text{dR}}^1(\mathbb{P}_k^1) \cong (H_{\text{dR}}^1(X))^G \cong \epsilon_G H_{\text{dR}}^1(X)$  and therefore that  $\epsilon_G x = 0$  for all  $x$ .

If  $S$  and  $T$  are distinct subsets then  $\epsilon_S \epsilon_T = 2^{2-2n} \sum h_s h_t$  where the sum ranges over all  $h_s \in H_S$  and  $h_t \in H_T$ . For each  $g \in G$ , we see that  $gh_s^{-1} \in H_T$  for half of the values of  $h_s \in H_S$ . So  $g$  appears  $2^{n-2}$  times in  $\sum h_s h_t$ . Thus,  $2^{2n-2} \epsilon_S \epsilon_T = 2^{n-2} \sum_{g \in G} g$  and we obtain that  $\epsilon_S \epsilon_T = \epsilon_G$ . Similarly, one can show for all subsets  $S$  that  $\epsilon_S \epsilon_S = \epsilon_S$  and  $\epsilon_S \epsilon_G = \epsilon_G$ .

We construct an explicit homomorphism  $\gamma$  from  $\oplus_S H_{\text{dR}}^1(C_S)$  to  $H_{\text{dR}}^1(X)$ :

$$\gamma(x_1, x_2, \dots, x_N) = \sum_{i=1}^N x_i.$$

If  $\psi$  is the homomorphism from  $H_{\text{dR}}^1(X)$  to  $\oplus_S H_{\text{dR}}^1(C_S)$  given by

$$\psi(y) = (\epsilon_1 y, \epsilon_2 y, \dots, \epsilon_N y)$$

then one can check that  $\psi \circ \gamma = \gamma \circ \psi = \text{Id}$ . Thus  $\gamma$  is an isomorphism of  $k$ -vector spaces. In fact,  $\gamma$  is a  $k[G]$ -module isomorphism since every  $g \in G$  commutes with  $\epsilon_S$  and thus with  $\gamma$ .  $\square$

The following corollary will be used throughout the remainder of the paper.

**Corollary 2.** *Suppose  $\text{char}(k) > 2$ . There is an isomorphism between  $\text{Jac}(X)[p]$  and  $\prod_S(\text{Jac}(C_S)[p])$  as group schemes where the product is taken over all non-empty  $S \subset \{1, \dots, n\}$ . In particular,  $\text{Jac}(X)$  and  $\prod_S(\text{Jac}(C_S))$  have the same  $p$ -rank and  $a$ -number.*

*Proof.* By Proposition 4, there is an isomorphism of  $k$ -vector spaces between  $H_{\text{dR}}^1(X)$  and  $\oplus_S H_{\text{dR}}^1(C_S)$ . By the functoriality of the Frobenius and Verschiebung morphisms,  $F$  and  $V$  commute with the action of  $g \in G$  and thus with the idempotents  $\epsilon_S$ . It follows that  $H_{\text{dR}}^1(X)$  and  $\oplus_S H_{\text{dR}}^1(C_S)$  are naturally isomorphic as  $k[V, F]$ -modules. Since  $X$  and  $C_S$  are smooth curves, [5, 3.11.2] implies that  $H_{\text{dR}}^1(\text{Jac}(X))$  and  $\oplus_S H_{\text{dR}}^1(\text{Jac}(C_S))$  are isomorphic as  $k[V, F]$ -modules. By [9, 5.11],  $H_{\text{dR}}^1(\text{Jac}(X))$  is canonically isomorphic to the contravariant Dieudonné module associated to  $\text{Jac}(X)[p]$ . Likewise,  $H_{\text{dR}}^1(\text{Jac}(C_S))$  is canonically isomorphic to the contravariant Dieudonné module associated to  $\text{Jac}(C_S)[p]$ . So the Dieudonné module of  $\text{Jac}(X)[p]$  is isomorphic to the direct sum of the Dieudonné modules of  $\text{Jac}(C_S)[p]$ . It follows, from the equivalence of categories between finite commutative group schemes over  $k$  and their contravariant Dieudonné modules, that the group schemes  $\text{Jac}(X)[p]$  and  $\prod_S(\text{Jac}(C_S)[p])$  are isomorphic.  $\square$

#### 4. Configurations of non-ordinary hyperelliptic curves

The results in this section will be used to find curves  $X$  having interesting  $p$ -power torsion, as measured in terms of invariants such as the  $p$ -rank and  $a$ -number. Corollary 2 shows that when a cover  $\phi : X \rightarrow \mathbb{P}^1$  corresponds to a point of  $H_{g,n}$  then such invariants for  $X$  can be determined by the corresponding invariants of its  $\mathbb{Z}/2$ -quotients. Since these quotients are all hyperelliptic, one can apply results of Yui [17]. The main difficulty is to control the  $p$ -torsion of all of the curves  $C_S$  in terms of the  $p$ -torsion of the curves  $C_i$ .

Let  $C$  be a smooth hyperelliptic curve of genus  $g$  defined over an algebraically closed field  $k$  of characteristic  $p > 2$ . Recall that  $C$  admits a  $\mathbb{Z}/2$ -Galois cover  $\phi_1 : C \rightarrow \mathbb{P}_k^1$  with  $2g + 2$  distinct branch points. Without loss of generality, we suppose  $\phi_1$  is branched at  $\infty$  and choose an equation for this cover of the form  $y^2 = f(x)$ , where  $f(x)$  is a polynomial of degree  $2g + 1$ . We denote the roots of  $f(x)$  by  $\{\lambda_1, \dots, \lambda_{2g+1}\}$ .

Denote by  $c_r$  the coefficient of  $x^r$  in the expansion of  $f(x)^{(p-1)/2}$ . Then

$$c_r = (-1)^{r-(p-1)/2} \sum \binom{(p-1)/2}{a_1} \dots \binom{(p-1)/2}{a_{2g+1}} \lambda_1^{a_1} \dots \lambda_{2g+1}^{a_{2g+1}} \quad (1)$$

where the sum ranges over all  $2g + 1$ -tuples  $(a_1, \dots, a_{2g+1})$  of integers such that  $0 \leq a_i \leq (p-1)/2$  for all  $i$  and  $\sum a_i = (2g+1)(p-1)/2 - r$ . Note that  $c_r$  can

be viewed as a polynomial in  $k[\lambda_1, \dots, \lambda_{2g+1}]$  which is homogeneous of degree  $(2g + 1)(p - 1)/2 - r$  and which is of degree  $(p - 1)/2$  in each variable.

Let  $A_g$  be the  $g \times g$  matrix whose  $ij$ th entry is  $c_{ip-j}$ . The determinant of  $A_g$  defines a polynomial in  $k[\lambda_1, \dots, \lambda_{2g+1}]$  which we denote by  $\text{Det}_g(\lambda_1, \dots, \lambda_{2g+1}) = \text{Det}_g(\vec{\lambda}_{2g+1})$ . This polynomial is of degree at most  $g(p - 1)/2$  in each  $\lambda_i$  and is homogeneous of total degree  $g^2(p - 1)/2$ . It is invariant under the action of  $S_{2g+1}$  on the variables  $\lambda_i$ . We denote by  $D_g \subset (\mathbb{A}_k^1)^{2g+1}$  the hypersurface of points  $\vec{\lambda}_{2g+1} = (\lambda_1, \dots, \lambda_{2g+1})$  for which  $\text{Det}_g(\vec{\lambda}_{2g+1}) = 0$ .

In [17], Yui gives the following characterization of non-ordinary hyperelliptic curves. Recall that  $\Delta$  is the weak diagonal consisting of points with at least two equal coordinates.

**Theorem 2.** (Yui [17]) *Suppose  $C$  is a smooth hyperelliptic curve of genus  $g$ . Then  $C$  is non-ordinary if and only if there is a  $\mathbb{Z}/2$ -Galois cover  $\phi : C \rightarrow \mathbb{P}_k^1$  branched at  $\infty$  and at  $2g + 1$  distinct points  $\lambda_i \in \mathbb{A}_k^1$  such that  $\vec{\lambda}_{2g+1} \in D_g$ .*

We now find some results on the geometry of the hypersurface  $D_g$  which will be used in Sections 5 and 6 to construct curves in  $\mathcal{H}_{g,n}$  whose  $p$ -torsion has prescribed invariants. In Lemma 4 and Lemma 5, we show that  $\text{Det}_g(\vec{\lambda}_{2g+1})$  is generically a polynomial of degree  $d = g(p - 1)/2$  in the variable  $\lambda_{2g+1}$  whose roots are not contained in  $\{\lambda_1, \dots, \lambda_{2g}\}$ . We expect for a generic choice of  $\lambda_1, \dots, \lambda_{2g}$  that this polynomial will have  $d$  distinct roots. Showing this seems to be related to the question of whether the hyperelliptic locus is transversal (in the strict geometric sense) to the locus  $V_{g,g-1}$  of nonordinary curves. In Proposition 5, we instead prove the weaker statement that this polynomial has at least  $(p - 1)/2$  distinct roots.

**Lemma 4.** *The determinant  $\text{Det}_g(\vec{\lambda}_{2g+1})$  is a polynomial of degree  $d = g(p - 1)/2$  in the variable  $\lambda_{2g+1}$ .*

*Proof.* As we observed above, the degree of  $\text{Det}_g(\vec{\lambda}_{2g+1})$  in  $\lambda_{2g+1}$  is at most  $d$ . We claim that the coefficient of  $\lambda_{2g+1}^d$  is a non-zero polynomial in  $k[\lambda_1, \dots, \lambda_{2g}]$ . In particular, one term of this polynomial is  $(-1)^{g(p-1)/2} \lambda_{2g+1}^d \prod_{i=1}^{2g} \lambda_i^{(g-[i/2])(p-1)/2}$ .

To see this, we note first from Equation 1 that the total degree of  $c_{gp-j}$  is  $(2g + 1)(p - 1)/2 - (gp - j) = (p - 1)/2 + (j - g)$ . So if  $j < g$  then  $\lambda_{2g+1}^{(p-1)/2}$  cannot appear in  $c_{gp-j}$ . Furthermore, the coefficient of  $\lambda_{2g+1}^{(p-1)/2}$  in  $c_{gp-g}$  is exactly  $(-1)^{(p-1)/2}$ . Because the degree of  $\lambda_{2g+1}$  in  $c_r$  is at most  $(p - 1)/2$  for all  $r$ , a monomial in  $\text{Det}_g(\vec{\lambda}_{2g+1})$  will be divisible by  $\lambda_{2g+1}^d$  only if it is the product of matrix entries which are each divisible by  $\lambda_{2g+1}^{(p-1)/2}$ . Thus  $c_{gp-g}$  is the only entry in the bottom row of  $A_g$  which contributes to the terms of  $\text{Det}_g(\vec{\lambda}_{2g+1})$  which are divisible by  $\lambda_{2g+1}^d$ .

Similarly, in the penultimate row of  $A_g$ , the total degree of  $c_{(g-1)p-j}$  will be  $3(p - 1)/2 + (j - (g - 1))$ . Therefore, if  $j < g - 1$  then  $(\lambda_1 \lambda_2 \lambda_{2g+1})^{(p-1)/2}$  cannot divide  $c_{(g-1)p-j}$ . Because the degree of  $\lambda_1$  for all  $c_r$  in  $A_g$  is at most  $(p - 1)/2$ , only the last two entries of the penultimate row contribute to the terms of  $\text{Det}_g(\vec{\lambda}_{2g+1})$

which are divisible by  $\lambda_1^{(g-1)(p-1)/2}$ . Also the coefficient of  $(\lambda_1\lambda_2\lambda_{2g+1})^{(p-1)/2}$  in  $c_{(g-1)p-(g-1)}$  is  $(-1)^{(p-1)/2}$ .

Continuing, we see that only terms which are on or above the diagonal can contribute to the desired term of  $\text{Det}_g(\vec{\lambda}_{2g+1})$ . It follows that the only term of  $\text{Det}_g(\vec{\lambda}_{2g+1})$  which involves the monomial  $\lambda_{2g+1}^d \prod_{i=1}^{2g} \lambda_i^{(g-\lceil i/2 \rceil)(p-1)/2}$  comes from the product of elements of the diagonal. The coefficient of this monomial is the product of  $g$  coefficients which each equal  $(-1)^{(p-1)/2}$ , so it is equal to  $(-1)^{g(p-1)/2}$ .  $\square$

**Lemma 5.** *The image of  $\text{Det}_g(\vec{\lambda}_{2g+1})$  in  $k[\lambda_1, \dots, \lambda_{2g+1}]/(\lambda_{2g+1} - \lambda_1)$  is non-constant.*

*Proof.* The proof is similar to that of Lemma 4. It is sufficient to show that at least one of the coefficients of  $\text{Det}_g(\lambda_1, \dots, \lambda_{2g}, \lambda_1)$  is non-zero. The coefficient of the monomial  $\lambda_1^{g(p-1)/2} \prod_{i=1}^{2g} \lambda_i^{(g-\lceil i/2 \rceil)(p-1)/2}$  is  $2(-1)^{g(p-1)/2}$  as this monomial appears exactly twice as the product of terms in the diagonal of the Hasse-Witt matrix and does not appear again in the expansion of the determinant.  $\square$

Suppose exactly two branch points of a smooth hyperelliptic cover specialize together. The resulting curve is singular and consists of a hyperelliptic curve  $C'$  of genus  $g - 1$  self-intersecting in a point. The geometric interpretation of the next lemma is that this singular curve will be ordinary if and only if  $C'$  is ordinary.

**Lemma 6.**  $\text{Det}_g(\lambda_1, \dots, \lambda_{2g-1}, 0, 0) = (-\lambda_1 \cdots \lambda_{2g-1})^{(p-1)/2} \text{Det}_{g-1}(\lambda_1, \dots, \lambda_{2g-1})$ .

*Proof.* Suppose  $\lambda_{2g} = \lambda_{2g+1} = 0$ . Then the only nonzero terms in the sum defining  $c_r$  are those where  $a_{2g} = a_{2g+1} = 0$ . If  $r = p - 1$ , then the only term in this sum that does not vanish is the one where  $a_i = (p - 1)/2$  for  $1 \leq i \leq 2g - 1$ . Thus  $c_{p-1} = (-\lambda_1 \cdots \lambda_{2g-1})^{(p-1)/2}$ . If  $r < p - 1$ , then all of the terms in the sum are zero, and thus  $c_r = 0$ . Suppose  $r > p - 1$  and  $r = ip - j$ . Then the term  $c_r$  occurring in the  $i$ th row and  $j$ th column of  $A_g$  equals the term  $c_{r-(p-1)}$  occurring in the  $(i - 1)$ st row and  $(j - 1)$ st column of  $A_{g-1}$ . By expanding the determinant along the first row, we see that  $\text{Det}_g(\lambda_1, \dots, \lambda_{2g-1}, 0, 0) = c_{p-1} \text{Det}(A_{g-1})$ .  $\square$

For fixed  $\vec{\lambda}_{2g} = (\lambda_1, \dots, \lambda_{2g}) \in (\mathbb{A}_k^1)^{2g}$ , denote by  $L(\vec{\lambda}_{2g})$  the line consisting of points  $(\lambda_1, \dots, \lambda_{2g}, \lambda_{2g+1}) \in (\mathbb{A}_k^1)^{2g+1}$  (where only the last coordinate varies). Generically, the intersection of  $L(\vec{\lambda}_{2g})$  and  $D_g$  consists of  $d = g(p - 1)/2$  points when counted with multiplicity. To see this, consider  $\text{Det}(\vec{\lambda}_{2g+1})$  as a polynomial in  $R[\lambda_{2g+1}]$  where  $R = k[\lambda_1, \dots, \lambda_{2g}]$ . The coefficient of  $\lambda_{2g+1}^d$  in  $\text{Det}(\vec{\lambda}_{2g+1})$  is non-zero in  $R$  by Lemma 4. Since  $k$  is an algebraically closed field, for any  $\vec{\lambda}_{2g} = (\lambda_1, \dots, \lambda_{2g})$  not in the Zariski closed set of  $(\mathbb{A}_k^1)^{2g}$  defined by this coefficient,  $\text{Det}(\vec{\lambda}_{2g+1})$  has degree  $d$  and thus  $d$  roots in  $k$  when counted with multiplicity. The next proposition gives a lower bound on the number of distinct roots.

**Proposition 5.** *Let  $U_g \subset (\mathbb{A}_k^1)^{2g}$  be the set of points  $(\lambda_1, \dots, \lambda_{2g})$  for which  $L(\vec{\lambda}_{2g})$  intersects  $D_g$  in at least  $(p - 1)/2$  non-zero distinct points of  $(\mathbb{A}_k^1)^{2g+1} \setminus \Delta$ . Then  $U_g$  is a nonempty Zariski open subset of  $(\mathbb{A}_k^1)^{2g}$ .*

*Proof.* The proof is by induction on  $g$ . A result of Igusa [4] states that there are exactly  $(p - 1)/2$  distinct values  $\lambda$  so that the elliptic curve branched at  $\{0, 1, \infty, \lambda\}$  is non-ordinary. It follows that the result is true when  $g = 1$ .

Suppose that  $U_{g-1}$  is a nonempty Zariski open subset of  $(\mathbb{A}_k^1)^{2g-2}$ . First we show that for a generic choice of  $(\lambda_1, \dots, \lambda_{2g})$  there are at least  $(p - 1)/2$  distinct choices of  $\lambda_{2g+1}$  so that  $\text{Det}_g(\lambda_1, \dots, \lambda_{2g}, \lambda_{2g+1}) = 0$ . It will suffice to construct a single choice of  $(\lambda_1, \dots, \lambda_{2g})$  for which this result holds, as the generic case will have at least as many distinct roots as any specialized case. It follows from Lemma 6 that for non-zero  $\lambda_3, \dots, \lambda_{2g}$ , the non-zero values of  $\lambda_{2g+1}$  so that  $\text{Det}_g(0, 0, \lambda_3, \dots, \lambda_{2g}, \lambda_{2g+1}) = 0$  and  $\text{Det}_{g-1}(\lambda_3, \dots, \lambda_{2g+1}) = 0$  are the same. By the inductive hypothesis, for the generic choice of  $(\lambda_3, \dots, \lambda_{2g})$  there are at least  $(p - 1)/2$  non-zero distinct values of  $\lambda_{2g+1}$  with this property.

Next we show that generically these  $(p - 1)/2$  intersection points of  $L(\vec{\lambda}_{2g})$  and  $D_g$  are not contained in  $\Delta$ . By Lemma 5 and by symmetry, for each  $1 \leq i \leq 2g$ , the value  $\lambda_i$  is a root of the polynomial  $\text{Det}_g(\lambda_1, \dots, \lambda_{2g+1}) \in R[\lambda_{2g+1}]$  only when  $(\lambda_1, \dots, \lambda_{2g})$  is in a Zariski closed subset of  $(\mathbb{A}^1)^{2g}$ . So for the generic choice of  $(\lambda_1, \dots, \lambda_{2g})$ , the root  $\lambda_{2g+1}$  will not be contained in  $\{\lambda_1, \dots, \lambda_{2g}\}$ . It follows that for the generic choice of  $(\lambda_1, \dots, \lambda_{2g})$  the line  $L(\vec{\lambda}_{2g})$  intersects  $D_g$  in at least  $(p - 1)/2$  non-zero distinct points of  $(\mathbb{A}_k^1)^{2g+1} \setminus \Delta$ . So  $U_g$  is a nonempty Zariski open subset of  $(\mathbb{A}_k^1)^{2g}$ .  $\square$

**Proposition 6.** *Let  $U_g \subseteq (\mathbb{A}_k^1)^{2g}$  be defined as in Proposition 5. Then we have that  $U_g \cap (D_{g-1} \times \mathbb{A}_k^1)$  has codimension 1 in  $(\mathbb{A}_k^1)^{2g}$ .*

*Proof.* Since  $D_{g-1} \times \mathbb{A}_k^1$  has codimension 1 in  $(\mathbb{A}^1)^{2g}$  and  $U_g$  is open by Proposition 5, it is sufficient to show that no component  $V$  of  $D_{g-1} \times \mathbb{A}_k^1$  is contained in the complement  $W_g$  of  $U_g$ . Note that the complement of  $U_g$  is a Zariski closed subset defined by equations which are each symmetric in the variables  $\lambda_1, \dots, \lambda_{2g}$ . On the other hand, any component  $V$  of  $D_{g-1} \times \mathbb{A}_k^1$  is defined by equations that do not involve  $\lambda_{2g}$ . Since the ideal of  $W_g$  is not contained in the ideal of  $V$ , it follows that  $V$  is not contained in  $W_g$ .  $\square$

### 5. Curves with prescribed $a$ -number

We now consider the  $a$ -number of Jacobians of curves with commuting involutions. Recall that the  $a$ -number,  $\dim_k \text{Hom}(\alpha_p, \text{Jac}(X))$ , of a  $k$ -curve  $X$  is an integer between 0 and  $g$ . Here  $\alpha_p$  is the kernel of Frobenius on  $\mathbb{G}_a$ . A generic curve is ordinary and thus has  $a$ -number equal to zero. A supersingular elliptic curve  $E$  has  $a$ -number equal to one and in this case there is a non-split exact sequence  $0 \rightarrow \alpha_p \rightarrow E[p] \rightarrow \alpha_p \rightarrow 0$ . There is a unique isomorphism type of group scheme for the  $p$ -torsion of a supersingular elliptic curve, which we denote  $M$ . In this section we construct curves  $X$  so that  $\text{Jac}(X)[p]$  contains multiple copies of the group scheme  $M$  and thus has large  $a$ -number.

Let  $T_{g,a}$  denote the sublocus of  $\overline{\mathcal{M}}_g$  consisting of curves of genus  $g$  with  $a$ -number at least  $a$ . The codimension of  $T_{g,a}$  in  $\mathcal{M}_g$  is at least  $a$  since  $T_{g,a} \subseteq V_{g,g-a}$ . It is not known whether (for all  $g$  and all  $0 \leq a \leq g$ ) there exists a curve of

genus  $g$  with  $a$ -number equal to  $a$ . The results in this section give some evidence for a positive answer to this question.

We note that these results can be viewed as a generalization of [11, Section 5]. In that paper, Oort considers curves  $X$  of genus  $g = 3$  with a group action by  $G = (\mathbb{Z}/2)^2$  so that the three  $\mathbb{Z}/2$ -quotients of  $X$  are all elliptic curves. He shows that there exist (nonhyperelliptic) curves of genus 3 with  $a$ -number 3 for all primes  $p \geq 3$  as well as hyperelliptic supersingular curves of genus 3 with  $a$ -number 3 for all  $p \equiv 3 \pmod 4$ .

**Lemma 7.** *The generic geometric point of the hyperelliptic locus  $\mathcal{H}_g$  has  $a$ -number equal to 0. The non-ordinary locus has codimension one in  $\mathcal{H}_g$  and its generic geometric point has  $a$ -number 1 and  $p$ -rank  $g - 1$ .*

*Proof.* This is immediate from Theorem 1 and the fact that a curve with  $p$ -rank  $g - 1$  has  $a$ -number 1.  $\square$

The next theorem will lead immediately to Corollary 3 which is the main result in this paper on the  $a$ -number of curves.

**Theorem 3.** *Suppose  $n \geq 2$  and  $p \geq 2n + 1$ . Suppose  $g$  is such that  $g \equiv 1 \pmod{2^{n-2}}$  and  $g \geq (n - 1)2^{n-2} + 1$ . There exists a family of smooth curves  $X$  of genus  $g$  of dimension at least  $(g + 2^n - 1)/2^{n-2} - 3 - n$  so that  $\text{Jac}(X)[p]$  contains the group scheme  $M^n$ .*

For the proof of Theorem 3, we will construct a fibre product  $\phi : X \rightarrow \mathbb{P}^1$  of  $n$  hyperelliptic covers  $\phi_i$  so that the disjoint union of any two of the branch loci  $B_i$  will consist of exactly two points. It follows that the curves  $C_{i,j}$  will have genus zero.

*Proof.* Write  $g = 1 + \ell 2^{n-2}$ . If  $\ell \not\equiv n \pmod 2$ , let  $g_1 = (\ell + 3 - n)/2$ . Note that  $g_1 \geq 1$ . By Proposition 5 and Lemma 7, as long as  $n \leq (p - 1)/2$ , there is a Zariski open subset  $U_{g_1}$  of  $(\mathbb{A}_k^1)^{2g_1}$  with the following property: there are at least  $n$  choices  $\eta_1, \dots, \eta_n$  for  $\lambda_{2g_1+1}$  such that the corresponding hyperelliptic curve  $C_i$  is non-ordinary. By Theorem 1, after replacing  $U_{g_1}$  with a smaller Zariski open subset of  $(\mathbb{A}_k^1)^{2g_1}$ , we can further suppose that the curves  $C_1, \dots, C_n$  will have  $p$ -rank  $g_1 - 1$ . Thus  $\text{Jac}(C_i)[p]$  contains  $M$ .

Let  $\phi_i : C_i \rightarrow \mathbb{P}^1$  for  $1 \leq i \leq n$  be the hyperelliptic  $U_{g_1}$ -curves corresponding to these choices. Let  $\phi : X \rightarrow \mathbb{P}^1$  be the normalization of the fibre product of the covers  $\phi_i$ . Note that  $X$  is branched at  $B = \{\infty, \lambda_1, \dots, \lambda_{2g_1}, \eta_1, \dots, \eta_n\}$ . By Proposition 3, the genus of  $X$  will be  $2^{n-2}(2g_1 + 1 + n) - 2^n + 1 = g$ . By Corollary 2,  $\text{Jac}(X)[p]$  contains  $\bigoplus_{i=1}^n \text{Jac}(C_i)[p]$  which contains  $\bigoplus_{i=1}^n M$ . The dimension of this family of curves is  $2g_1 - 2 = |B| - 3 - n$  which equals  $(g + 2^n - 1)/2^{n-2} - 3 - n$ . Note that the  $p$ -rank of  $X$  is at least  $n(g_1 - 1)$ .

Alternatively, if  $\ell \equiv n \pmod 2$ , let  $g_1 = (\ell + 2 - n)/2$  and note  $g_1 \geq 1$ . By Proposition 6, the locus  $U_{g_1+1} \cap (D_{g_1} \times \mathbb{A}_\ell^1)$  has codimension 1 in  $(\mathbb{A}_k^1)^{2g_1+2}$ . In other words, as long as  $n \leq 1 + (p - 1)/2$ , for any  $(\lambda_1, \dots, \lambda_{2g_1+2})$  in a codimension 1 subset  $Z$  of  $(\mathbb{A}_k^1)^{2g_1+2}$ , it is true that  $(\lambda_1, \dots, \lambda_{2g_1+1}) \in D_{g_1}$  and there are at least  $n - 1$  choices  $\eta_i$  of  $\lambda_{2g_1+3}$  with  $(\lambda_1, \dots, \lambda_{2g_1+3}) \in D_{g_1+1}$ . Let  $\phi_n : C_n \rightarrow \mathbb{P}^1$

be the hyperelliptic cover branched at  $(\lambda_1, \dots, \lambda_{2g_1+1})$ . For  $1 \leq i \leq n - 1$ , let  $\phi_i : C_i \rightarrow \mathbb{P}^1$  be the hyperelliptic cover branched at  $(\lambda_1, \dots, \lambda_{2g_1+2}, \eta_i)$ . Then  $C_n$  has genus  $g_1$  and  $C_i$  has genus  $g_1 + 1$  for  $1 \leq i \leq n - 1$ . By Theorem 1, after restricting to a Zariski open subset of  $Z$ , we can further suppose that  $C_n$  (resp.  $C_i$ ) has  $p$ -rank  $g_1 - 1$  (resp.  $g_1$ ). Thus  $\text{Jac}(C_i)[p]$  contains  $M$  for  $1 \leq i \leq n$ .

Let  $\phi : X \rightarrow \mathbb{P}^1$  be the normalization of the fibre product of  $\phi_i$  for  $1 \leq i \leq n$ . Note that  $\phi$  is branched at  $B = \{\infty, \lambda_1, \dots, \lambda_{2g_1+2}, \eta_1, \dots, \eta_{n-1}\}$ . As above,  $X$  has genus  $2^{n-2}(2g_1 + 2 + n) - 2^n + 1 = g$  and  $\text{Jac}(X)[p]$  contains  $M^n$ . By Proposition 6, the locus  $Z$  has dimension  $2g_1 + 1$ . The corresponding family of curves has dimension  $2g_1 - 1 = |B| - 3 - n$  which again equals  $(g + 2^n - 1)/2^{n-2} - 3 - n$ . Note that the  $p$ -rank of  $X$  is at least  $ng_1 - 1$ .  $\square$

**Corollary 3.** *Suppose  $n \geq 2$  and  $p \geq 2n + 1$ . Suppose  $g$  is such that  $\mathcal{H}_{g,n}$  is non-empty of dimension at least  $n + 1$ . Then the intersection  $\mathcal{H}_{g,n} \cap T_{g,n}$  has codimension at most  $n$  in  $\mathcal{H}_{g,n}$ . In particular, there exists a smooth curve of genus  $g$  with  $a$ -number at least  $n$ .*

*Proof.* By Corollary 1, the condition that  $\mathcal{H}_{g,n}$  is non-empty is equivalent to  $g \equiv 1 \pmod{2^{n-2}}$  and the condition that  $\mathcal{H}_{g,n}$  has dimension at least  $n + 1$  is equivalent to  $g \geq (n - 1)2^{n-2} + 1$ . The family constructed in Theorem 3 has dimension  $(g + 2^n - 1)/2^{n-2} - 3 - n$  and thus codimension  $n$  in  $\mathcal{H}_{g,n}$ . For any fibre  $X$  in this family,  $\text{Jac}(X)[p]$  contains  $M^n$  and so  $X$  has  $a$ -number at least  $n$ . So this family is contained in  $\mathcal{H}_{g,n} \cap T_{g,n}$ .  $\square$

When  $n = 2$  or  $n = 3$ , then the curves found in Theorem 3 are in fact hyperelliptic.

**Corollary 4.** *Suppose  $g \geq 2$  and  $p \geq 5$ . There exists a  $(g - 2)$ -dimensional family of smooth hyperelliptic curves of genus  $g$  whose fibres have  $a$ -number 2 and  $p$ -rank  $g - 2$ .*

The family in Corollary 4 has codimension 2 in  $\mathcal{H}_{g,2}$ .

*Proof.* This follows immediately from Theorem 3 once we show that the curve  $X$  is hyperelliptic when  $n = 2$ . If  $g$  is even, note that the branch loci of  $\phi_1$  and  $\phi_2$  differ in only one point. The third quotient  $C_{1,2}$  of  $X$  by  $\mathbb{Z}/2$  is branched at only two points  $\eta_1$  and  $\eta_2$ . So the cover  $X \rightarrow C_{1,2}$  is hyperelliptic. Likewise, if  $g$  is odd, then the third quotient  $C_{1,2}$  of  $X$  by  $\mathbb{Z}/2$  is branched at only two points  $\lambda_{2g_1+2}$  and  $\eta_1$  so the cover  $X \rightarrow C_{1,2}$  is hyperelliptic. In both cases,  $\text{Jac}(X)[p] \simeq \text{Jac}(C_1)[p] \oplus \text{Jac}(C_2)[p]$  and so the fibres of  $X$  have  $a$ -number 2 and  $p$ -rank  $g - 2$ .  $\square$

**Corollary 5.** *Suppose  $g \geq 5$  is odd and  $p \geq 7$ . There exists a  $(g - 5)/2$ -dimensional family of smooth hyperelliptic curves  $X$  of genus  $g$  so that  $\text{Jac}(X)[p]$  contains  $M^3$  and thus has  $a$ -number at least 3.*

The family in Corollary 5 has codimension 3 in  $\mathcal{H}_{g,3}$ .

*Proof.* It is sufficient to show that the fibres of the family constructed in Theorem 3 are hyperelliptic when  $n = 3$ . In both cases of the construction, if  $S = \{1, 2\}$ ,

$\{1, 3\}$ , or  $\{2, 3\}$ , then the quotient  $C_S \rightarrow \mathbb{P}^1$  of  $X$  by  $H_S \simeq (\mathbb{Z}/2)^2$  is branched at only two points and so  $C_S$  has genus 0. Consider the quotient  $X'$  of  $X$  by the subgroup  $H' \simeq \mathbb{Z}/2$  generated by  $h = (1, 1, 1)$ . Note that  $X'$  dominates  $C_S$  if  $S = \{1, 2\}$ ,  $\{1, 3\}$ , or  $\{2, 3\}$ , since  $h \in H_S$ . It follows that  $X'$  has genus zero since  $X' \rightarrow \mathbb{P}^1$  is a  $(\mathbb{Z}/2)^2$ -cover having three  $\mathbb{Z}/2$ -quotients of genus zero. It follows that  $X \rightarrow X'$  is hyperelliptic.  $\square$

*Remark 1.* One would like to strengthen Corollary 5 by producing curves with  $a$ -number exactly 3. The difficulty is to determine the  $a$ -number of  $C_{\{1,2,3\}}$ . For example, to construct a curve of genus  $g = 5$  and  $a$ -number exactly 3 with this method, one would need to guarantee that there are supersingular values  $\lambda_1, \lambda_2$  and  $\lambda_3$  so that the hyperelliptic curve of genus two branched at  $\{0, 1, \infty, \lambda_1, \lambda_2, \lambda_3\}$  is ordinary.

*Remark 2.* In the above results, some restriction on  $p$  is unavoidable. By Proposition 3.1 of [13], there does not exist a hyperelliptic curve of genus 2 and  $a$ -number 2 when  $p = 3$  or of genus 3 and  $a$ -number 3 when  $p = 3$  or 5. Also, there does not exist a hyperelliptic curve with  $a$ -number 4 when  $g = 4$  and  $p = 3, 5$  or when  $g = 5$  and  $p = 3$ .

We now produce curves of every genus with  $a$ -number at least 4 using this method. (One can also produce curves of every genus with  $a$ -number at least 3 and count the dimension of these families). The curves constructed in this way will most likely not be hyperelliptic. This makes it difficult to produce a curve of every genus with every possible  $a$ -number using induction and fibre products.

**Corollary 6.** *Suppose  $g \geq 7$  and  $p \geq 5$ . There exists a curve of genus  $g$  with  $a$ -number at least 4.*

*Proof.* If  $g$  is even, let  $g_1 = g/2$ . Note that  $g_1 - 2 \geq 2$ . From Corollary 4, there exists a hyperelliptic curve of genus  $g_1 - 2$  and  $a$ -number 2. Consider the corresponding hyperelliptic cover  $\phi_1$  branched at  $\{\lambda_1, \dots, \lambda_{2g_1-3}, \infty\}$ . Consider a hyperelliptic cover  $\phi_2$  branched at  $\{\eta_1, \dots, \eta_5, \infty\}$  which has  $a$ -number 2. After modifying  $\phi_2$  by an affine linear transformation, one can suppose that  $\{\eta_i\} \cap \{\lambda_i\}$  is empty. The cardinality of  $(B_1 \cup B_2) \setminus (B_1 \cap B_2)$  is  $(2g_1 - 2) + 6 - 2 = 2g_1 + 2$ . It follows from Proposition 3 that the fiber product of  $\phi_1$  and  $\phi_2$  yields a curve with genus  $(g_1 - 2) + g_1 + 2 = g$  and  $a$ -number at least 4.

If  $g$  is odd, let  $g_1 = (g - 1)/2$ . Note that  $g_1 - 1 \geq 2$ . By Corollary 4, there exists a hyperelliptic curve of genus  $g_1 - 1$  and  $a$ -number 2. Consider the corresponding hyperelliptic cover  $\phi_1$  branched at  $\{\lambda_1, \dots, \lambda_{2g_1-2}, 0, \infty\}$ . Consider a hyperelliptic cover  $\phi_2$  branched at  $\{\eta_1, \dots, \eta_4, 0, \infty\}$  which has  $a$ -number 2. After modifying  $\phi_2$  by a scalar transformation, one can suppose that  $\{\eta_i\} \cap \{\lambda_i\}$  is empty. The cardinality of  $(B_1 \cup B_2) \setminus (B_1 \cap B_2)$  is  $2g_1 + 6 - 4 = 2g_1 + 2$ . It follows from Proposition 3 that the fiber product of  $\phi_1$  and  $\phi_2$  yields a curve with genus  $(g_1 - 1) + g_1 + 2 = g$  and  $a$ -number at least 4.  $\square$

## 6. Curves with prescribed $p$ -torsion

The methods of the previous sections can also be used to construct Jacobians whose  $p$ -torsion contains group schemes other than  $\mu_p$  or  $\alpha_p$ . In this section, we illustrate this for two particular isomorphism types of group scheme, namely the  $p$ -torsion of a supersingular abelian surface which is not superspecial and of a supersingular abelian variety of dimension 3 with  $a$ -number 1.

Section 3 allows one to describe the  $p$ -torsion of the Jacobian of a curve  $X$  which corresponds to a point of  $\mathcal{H}_{g,n}$ . Specifically, Proposition 4 states that  $\text{Jac}(X)[p]$  is the direct sum of  $\text{Jac}(C_S)[p]$  where  $C_S$  is the quotient of  $X$  by  $H_S$  and  $S$  ranges over the  $2^n - 1$  nonempty subsets of  $\{1, \dots, n\}$ . With this method, it is only possible to construct Jacobians so that  $\text{Jac}(X)[p]$  is decomposable into (at least two) group schemes each of which occurs as the  $p$ -torsion of a hyperelliptic curve.

Via the  $p$ -rank, we have already considered the group scheme for the  $p$ -torsion of an ordinary elliptic curve, namely  $\mathbb{Z}/p \oplus \mu_p$ . Using the  $a$ -number, we have already studied the group scheme  $M$  of the  $p$ -torsion of a supersingular elliptic curve.

Not many other group schemes are known to occur as the  $p$ -torsion of a hyperelliptic curve. There are four possibilities of group scheme which occur among curves of genus 2 (which are automatically hyperelliptic). The first three  $(\mathbb{Z}/p \oplus \mu_p)^2$ ,  $(\mathbb{Z}/p \oplus \mu_p) \oplus M$ , and  $M^2$  are decomposable. We will focus on the most interesting of the four, namely the group scheme  $N$  for the  $p$ -torsion of a supersingular abelian surface which is not superspecial. A curve  $X$  with  $\text{Jac}(X)[p] \simeq N$  has genus 2 and is thus hyperelliptic.

By [3, Example A.3.15], there is a filtration  $H_1 \subset H_2 \subset N$  where  $H_1 \simeq \alpha_p$ ,  $H_2/H_1 \simeq \alpha_p \oplus \alpha_p$  and  $N/H_2 \simeq \alpha_p$ . Moreover, the kernel  $G_1$  of Frobenius and the kernel  $G_2$  of Verschiebung are contained in  $H_2$  and there is an exact sequence  $0 \rightarrow H_1 \rightarrow G_1 \oplus G_2 \rightarrow H_2 \rightarrow 0$ .

The group scheme  $N$  is perhaps easier to describe in terms of its covariant Dieudonné module. Consider the non-commutative ring  $E = W(k)[F, V]$  with the Frobenius automorphism  $\sigma : W(k) \rightarrow W(k)$  and the relations  $FV = VF = p$  and  $F\lambda = \lambda^\sigma F$  and  $\lambda V = V\lambda^\sigma$  for all  $\lambda \in W(k)$ . Recall that there is an equivalence of categories between finite commutative group schemes  $\mathbb{G}$  over  $k$  (with order  $p^r$ ) and finite left  $E$ -modules  $D(\mathbb{G})$  (having length  $r$  as a  $W(k)$ -module), see for example [3, A.5]. By [3, Example A.5.1-5.4],  $D(\mu_p) = k/k(V, 1 - F)$ ,  $D(\alpha_p) = k/k(F, V)$ , and  $D(N) = k/k(F^3, V^3, F^2 - V^2)$ .

The  $p$ -rank of a curve  $X$  with  $\text{Jac}(X)[p] \simeq N$  is zero. To see this, note that  $\text{Hom}(\mu_p, N) = 0$  or that  $F$  and  $V$  are both nilpotent on  $D(N)$ . The  $a$ -number of a curve  $X$  with  $\text{Jac}(X)[p] \simeq N$  is one. (It is at least one since the  $p$ -rank is 0 and at most one since the abelian surface is not superspecial.) This also follows from the structure of the group scheme or from the fact that  $N[F] \cap N[V] = H_1 \simeq \alpha_p$ .

**Lemma 8.** *There is a one-dimensional family of smooth curves  $X$  of genus two with  $\text{Jac}(X)[p] \simeq N$ .*

*Proof.* The dimension in  $\mathcal{A}_2$  of supersingular (resp. superspecial) abelian surfaces is one (resp. zero). It follows that the locus of abelian surfaces with  $p$ -torsion  $N$  is

exactly one. The generic point of this one-dimensional family must be in the image of the Torelli morphism since  $\overline{\mathcal{M}}_2$  and  $\overline{\mathcal{A}}_2$  have the same dimension. So there is a one-dimensional family of curves of genus two with  $p$ -rank 0 and  $a$ -number 1. The fibres of this family are all smooth since the family cannot intersect either of the boundary components  $\Delta_0$  or  $\Delta_1$ .  $\square$

**Lemma 9.** *There exists a one-dimensional family of smooth hyperelliptic curves  $X$  of genus 3 with  $\text{Jac}(X)[p] \simeq N \oplus (\mathbb{Z}/p \oplus \mu_p)$ .*

*Proof.* By Lemma 8, there is a one-dimensional family of smooth curves  $X$  of genus two with  $\text{Jac}(X)[p] \simeq N$ . This yields a family of hyperelliptic covers of  $\mathbb{P}^1$  branched at six points. For some subset of four of these points, the family of elliptic curves branched at these points must have varying  $j$ -invariant and so its fibres are generically ordinary. The fibre product of these two families of covers yields a family of smooth hyperelliptic curves of genus 3 with  $p$ -torsion  $N \oplus (\mathbb{Z}/p \oplus \mu_p)$  by Corollary 2.  $\square$

The following proposition will be used to generalize Lemma 9 for  $g \geq 4$ .

**Proposition 7.** *Suppose there exists an  $r$ -dimensional family of smooth hyperelliptic curves  $C$  of genus  $g'$  with  $\text{Jac}(C)[p] \simeq \mathbb{G}$  for some group scheme  $\mathbb{G}$ . Suppose  $s \geq 1$  and  $g = 2g' - 1 + s$ . Then there exists an  $(r + s)$ -dimensional family of smooth curves  $X$  in  $\mathcal{H}_{g,2}$  so that  $\text{Jac}(X)[p]$  contains  $\mathbb{G}$ .*

*Proof.* For each curve  $C$  in the original family with  $\text{Jac}(C)[p] \simeq \mathbb{G}$  and branch locus  $B_0 = \{\lambda_1, \dots, \lambda_{2g'+2}\}$ , we will construct an  $s$ -dimensional family of smooth curves  $X$  so that  $\text{Jac}(X)[p]$  contains  $\mathbb{G}$ . By Proposition 3 and Lemma 2, it will suffice to construct hyperelliptic curves  $C_1$  and  $C_2$  whose branch loci  $B_1$  and  $B_2$  are of even cardinality with  $|B_1 \cap B_2| = s$  and  $B_0 = (B_1 \cup B_2) \setminus (B_1 \cap B_2)$ .

If  $s = 2m$  is even, then  $B_1 = B_0 \cup \{\eta_1, \dots, \eta_{2m}\}$  and  $B_2 = \{\eta_1, \dots, \eta_{2m}\}$  satisfy these restrictions and there are  $2m = s$  choices for the points  $\eta_i$ . Similarly, if  $s = 2m + 1$  is odd, then we can set  $B_1 = \{\lambda_1, \dots, \lambda_{2g'+1}, \eta_1, \dots, \eta_{2m+1}\}$  and  $B_2 = \{\lambda_{2g'+2}, \eta_1, \dots, \eta_{2m+1}\}$  satisfy these restrictions. There are  $2m + 1 = s$  choices for  $\eta_i$ . The Jacobian of the normalized fibre product  $X$  of  $C_1$  and  $C_2$  contains  $\text{Jac}(C)$ .  $\square$

This is the main result of the section.

**Corollary 7.** *Let  $N$  be the  $p$ -torsion of a supersingular abelian surface which is not superspecial. For all  $g \geq 2$ , there exists a smooth hyperelliptic curve  $X$  so that  $\text{Jac}(X)[p]$  contains  $N$ .*

*Proof.* The statement will follow from induction. Assume for all  $g'$  such that  $2^n \leq g' < 2^{n+1}$  that there exists a smooth hyperelliptic curve  $X_{g'}$  so that  $\text{Jac}(X_{g'})[p]$  contains  $N$ . This is true for  $n = 1$  by Lemma 8 and Lemma 9. If  $2^{n+1} \leq g < 2^{n+2}$ , then  $g = 2g'$  or  $g = 2g' + 1$  for some  $g'$  such that  $2^n \leq g' < 2^{n+1}$ . Using Proposition 7 with  $s = 1$  or  $s = 2$  allows one to construct a curve  $X_g$  of genus  $g$  so that  $\text{Jac}(X_g)[p]$  contains  $N$ . If  $s = 1$  or  $s = 2$  in Proposition 7, then  $B_2$  consists of exactly two points so  $X_g$  is also hyperelliptic.  $\square$

Similarly, one can consider the group scheme  $Q$  of the  $p$ -torsion of a supersingular abelian variety of dimension three with  $a$ -number 1. A curve  $X$  with  $\text{Jac}(X)[p] = Q$  has  $p$ -rank 0. Also,  $D(Q) = k[F, V]/k(F^4, V^4, F^3 - V^3)$ . The restriction on  $g$  in the next corollary could be removed if there exists a smooth hyperelliptic curve  $X$  of genus 4 so that  $\text{Jac}(X)[p]$  contains  $Q$ .

**Corollary 8.** *Let  $Q$  be the  $p$ -torsion of a supersingular abelian variety of dimension three with  $a$ -number 1. Suppose  $g \geq 3$  is not a power of two. Then there exists a smooth hyperelliptic curve  $X$  of genus  $g$  so that  $\text{Jac}(X)[p]$  contains  $Q$ .*

*Proof.* The proof parallels that of Corollary 7. One starts with the supersingular hyperelliptic curve  $X$  of genus 3 and  $a$ -number 1 (and thus  $\text{Jac}(X)[p] \simeq Q$ ) from [11] and works inductively using Proposition 7.  $\square$

It is natural to ask whether Corollary 7 could be strengthened to state that  $\text{Jac}(X)[p] \simeq N \oplus (\mathbb{Z}/p \oplus \mu_p)^{g-2}$ . This raises the following geometric question.

*Question 1.* Given any choice of  $\Lambda = \{\lambda_1, \dots, \lambda_{2r}\}$ , does there exist  $\mu \in \mathbb{A}_k^1 - \Lambda$  so that the hyperelliptic curve branched at  $\{\lambda_1, \dots, \lambda_{2r}, \mu, \infty\}$  is ordinary?

For a generic choice of  $\Lambda$ , the answer to Question 1 is yes by Lemma 4. This question will have an affirmative answer if the hypersurface  $D_r$  does not contain any coordinate line  $L(\tilde{\lambda}_{2r})$ . The question is equivalent to asking whether, given a hyperelliptic cover  $\phi : X \rightarrow \mathbb{P}_k^1$ , it is always possible to deform  $X$  to an ordinary curve by moving only one branch point.

An affirmative answer to Question 1 would allow one to strengthen Proposition 7 to state that  $\text{Jac}(X)[p] \simeq \mathbb{G} \oplus (\mathbb{Z}/p \oplus \mu_p)^{g-1+s}$ . This is because the curves  $C_1$  and  $C_2$  in the proof can be generically chosen to be ordinary. So an affirmative answer to Question 1 would imply that for all  $g \geq 4$  there exists a smooth hyperelliptic curve  $X$  with  $\text{Jac}(X)[p] \simeq N \oplus (\mathbb{Z}/p \oplus \mu_p)^{g-2}$ . If this is true, then  $\text{Jac}(X)[p] \simeq N \oplus (\mathbb{Z}/p \oplus \mu_p)^{g-2}$  when  $X$  is the generic geometric point of  $\mathcal{H}_g \cap V_{g, g-2}$ .

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