Problems of the Month
Gettysburg College Mathematics Department
September, 2005

Details: Solutions should be submitted to the icosahedron in the math department by 5 pm on September 30th, at which point the math department faculty will grade the solutions. You are encouraged to work together, and submissions from groups will be accepted. If you have any questions, or would like to suggest problems of your own, please contact Darren Glass in Office 215E or at dglass@gettysburg.edu.

S1. An absent minded mathematician has six bills to pay. To do this, he writes six checks and addresses six envelopes but then randomly put one check into each envelope before mailing them. What is the probability that exactly half of the checks arrived at their correct destination?

S2. A sequence of rational numbers is defined by setting $a_1 = \frac{1}{2}, a_2 = \frac{5}{7},$ and in general letting the denominator of $a_n$ be given as the sum of the numerator and the denominator of $a_{n-1}$ and letting the numerator of $a_n$ be given by the sum of the denominators of $a_n$ and $a_{n-1}$. For example, this sequence starts:

$$
\frac{1}{2}, \frac{5}{3}, \frac{11}{8}, \frac{27}{19}, \frac{65}{46}, \ldots
$$

Find the limit of this sequence.

S3. A cube of cheese of size 3x3x3 is divided into 27 small cubes of size 1x1x1. A mouse eats one small cube each day and an adjacent small cube (one which shares a face) the next day. Can the mouse eat the center small cube on the last day?

S4. (From Mathematics Magazine) Show that $P_4(x) = x^4 + x^3 - x^2 - x + 1$ has at least one nonreal root. Show that $P_5(x) = x^5 + x^4 - x^3 - x^2 + x + 1$ has at least one nonreal root. In general, $P_n(x)$ is the polynomial whose coefficients “alternate in pairs”, starting with 1, 1, -1, -1, 1, 1, -1, -1, 1, 1, -1… until the end of the polynomial:

$$P_n(x) = x^n + x^{n-1} - x^{n-2} - x^{n-3} + x^{n-4} + x^{n-5} - x^{n-6} \ldots \pm x \pm 1$$

where the signs of the last two terms will depend on the value of $n$. Prove that $P_n(x)$ has at least one nonreal root.