Polynomial Time Reducibility

A

\[ w \]

Poly time TM

starts on input \( w \)

halts with \( f(w) \)

on the tape

\[ A \leq_p B \]
If $A \leq_p B$ and $B \in p$
then $A \in p$. 
3-SAT

CNF - conjunctive normal form

\((x \lor \overline{y} \lor z) \land (\text{...})\) \(\land (\text{...})\)

Any Boolean formula can be expressed in CNF

3 CNF

Each term has exactly 3 literals (e.g. \(x, \overline{x}\))

Term \((x_1 \lor x_2 \lor x_3)\)

3 SAT = \(\{\phi | \phi \text{ is a satisfiable 3 CNF formula}\}\)

CLIQUE = \(\{\langle G, k \rangle | \text{graph } G \text{ contains a } k \text{-clique}\}\)
$3SAT \leq_p CLIQUE$

$\phi = (a_1 \lor b_1 \lor c_1) \land (a_2 \lor b_2 \lor c_2) \land \ldots \land (a_k \lor b_k \lor c_k)$

Construct $G$

$k$ groups of 3 nodes

one node for each appearance of a literal in $\phi$

Edges between every pair except

1. nodes in the same group
2. nodes w/ contradictory labels (literals)
   eg $x, \overline{x}$
\[
\phi = (x_1 \lor x_1 \lor x_2) \land (\overline{x}_1 \lor \overline{x}_2 \lor \overline{x}_2) \\
\land (\overline{x}_1 \lor x_2 \lor x_2)
\]

\[x_2 = \top \]
\[x_1 = \bot \]
\[x_i = \top \]
\[x_i = \bot \]

\(\phi\) is satisfiable iff \(G\) has a \(k\)-clique (\(k = \#\) of terms in \(\phi\))

I. Suppose \(\phi\) is satisfiable.
   So each term will have at least one true literal.

   Consider the nodes w/ the labels from those true literals
   These form a clique in \(G\)

II. Suppose \(G\) has a \(k\)-clique
   - no two of the clique’s nodes are in the same group.

   - each group contains exactly one of the clique’s nodes
   - assign values to variables \(s.t.\) the literals are true
   - that assignment satisfies \(\phi\).
A language is NPC if two conditions are true:
1. The language is in NP.
2. Every language in NP is poly-time reducible to it.

If \( B \) is NPC and \( B \in P \), then \( P = NP \).

If \( B \) is NPC and \( B \leq_p C \) for \( C \) in NP, then \( C \) is NPC.
Cook–Levin Thm:
SAT is NP-Complete.