Reducibility

Problem $A$ reduces to problem $B$

If $A$ is reducible to problem $B$ and $B$ is decidable, then $A$ is decidable.

If $A$ is reducible to $B$ and $A$ is not decidable, then $B$ must be undecidable.
$\text{HALT}_{TM} = \{<m,w>| M \text{ is a TM and } M \text{ halts on input } w\}$

$\text{HALT}_{TM}$ is undecidable:
proof (by contradiction)
Suppose $\text{HALT}_{TM}$ is decidable
and TM $R$ decides it.

Construct $S$ to decide $A_{\text{TM}}$.

$S$: on input $<m,w>$
1. Run $R$ on $<m,w>$
2. If $R$ rejects, reject
3. If $R$ accepts, simulate $M$ on $w$ until $M$ halts.
4. If $M$ accepts, accept
   otherwise, reject
$E_{TM} = \{ \langle m \rangle \mid M \text{ is a TM and } L(m) = \emptyset \}$

$E_{TM}$ is undecidable

Proof (by contradiction)

Suppose it isn't. So TM R decides $E_{TM}$.

Given M, w create TM M'

M' = on input x
1. if x ≠ w, reject.
2. if x = w, run M on w and accept if M accepts.

$L(M') = \{ w \} \text{ if M accepts } w$
$L(M') = \emptyset \text{ otherwise.}$

$L(M') \neq \emptyset \text{ if M accepts } w.$

Construct S that decides $A_{TM}$.
S = on input $\langle M, w \rangle$
1. Use M, w to build M'
2. Run R on M'
3. if R accepts, reject.
   if R rejects, accept.
\[ EQ_{TM} = \{ \langle m_1, m_2 \rangle \mid \text{TM's } m_1 \text{ and } m_2 \text{ and } L(m_1) = L(m_2) \} \]

Reduction from \( E_{TM} \)

\( E_{TM} \)

\( \langle m \rangle \)

\( \langle m, N \rangle \)

\( EQ_{TM} \)

\( \text{acc} \)

\( \text{rej} \)

\( \text{rej} \)

\( \text{accept} \)

\( \text{reject} \)

\( TM \text{ N rejects everything} \)

\( L(N) = \emptyset \)
not Rec

CSG
\[\alpha A \beta \Rightarrow \alpha X \beta\]

A is a var
\(\alpha, \beta, X\) are strings
\(X \neq \varepsilon\)