Some languages are not Turing recognizable.

Proof:

Set of all Turing Machines is a subset of finite length strings so is countable.

Set of all languages is uncountable.

Suppose it is countable

\[
\begin{array}{c|cccccc}
\text{strings} & s_1 & s_2 & s_3 & s_4 & s_5 & \ldots \\
L_1 & \checkmark & x & \checkmark & \checkmark & x & \ldots \\
L_2 & \checkmark & \checkmark & x & \checkmark & x & \ldots \\
L_3 & \checkmark & x & \checkmark & x & \checkmark & \ldots \\
L_4 & x & \checkmark & \checkmark & \checkmark & x & \ldots \\
\vdots & & & & & &
\end{array}
\]

Construct \( L_n \) that is not in the list.

\( L_n \)'s characteristic sequence

\[
\begin{align*}
S_i \in L_n & \quad \text{if} \quad s_i \notin L_i \\
S_i \notin L_n & \quad \text{if} \quad s_i \in L_i
\end{align*}
\]

Therefore the set of lang. is uncountable.

Since TMs are countable and lang. are not, there must be some lang. not recognized by a TM.
$A_{\text{TM}} = \{ <M,w> | M \text{ is a TM that accepts } w \}$

Proof (by contradiction)

Suppose $A_{\text{TM}}$ is decidable
and $H$ is TM that decides it.

$H(<M,w>) =$ accept if $M$ accepts $w$
               reject if $M$ does not accept $w$.

Construct TM D that uses H as a subroutine.

$D$: on input $<M>$ where M is a TM
1. Run $H$ on input $<M,<M>>$
2. if $H$ accepts $\Rightarrow$ reject
   if $H$ rejects $\Rightarrow$ accept

Now run $D$ on itself:

$D(<D>) =$ accept if $D$ does not accept $<D>$
          reject if $D$ accepts $<D>$

which is a contradiction.

Therefore, $H$ cannot exist so $A_{\text{TM}}$ is not decidable.
A language is co-Turing Recognizable if its complement is Turing Rec.

A language is decidable iff it is Turing Rec. and Co-Turing Rec.

\[ A_m \] is not Turing Rec.
Reducibility

A reduces to B

if A is reducible to B and B is decidable then A is decidable.

if A is reducible to B and A is undecidable, then B is undecidable.