Problem Reduction

Convert one problem to another

A reduces to B
we can use solution of B
to solve A

if A is reducible to B
and B is decidable,
then A is decidable.

if A is undecidable and
reducible to B then
B is undecidable.
\[ \text{HALT}_{TM} = \{ \langle m, w \rangle \mid M \text{ is a TM that halts on } w \} \]

\text{HALT}_{TM} \text{ is undecidable}

proof by contradiction

Suppose \text{HALT}_{TM} \text{ is decidable w/ TM R}

Construct a TM S to decide \text{A}_{TM} using R.

\[ S = \text{on input } \langle m, w \rangle \]

1. Run R on input \langle m, w \rangle.
2. If R rejects, reject.
3. If R accepts, we know M accepts w or M rejects w.
   - Simulate M on w.
     - If M accepts, accept.
     - Otherwise reject.

So S decides \text{A}_{TM}.

However, \text{A}_{TM} \text{ is undecidable so R does not exist.}

Reduced \text{A}_{TM} to \text{HALT}_{TM}
$$E_{TM} = \{\langle m \rangle \mid m \text{ is a TM and } L(m) = \emptyset \}$$

$E_{TM}$ is undecidable by reduction from $A_{TM}$

Proof by contradiction
Suppose $E_{TM}$ is decidable by T.M. $R$.

Given $m, w$ create $M'$

$M'$ on input $x$
1. if $x \neq w$ reject
2. if $x = w$ run $M$ on $w$ and accept if $M$ accepts

Construct $S$ to decide $A_{TM}$

$S$ on input $\langle m, w \rangle$
1. use description of $M$ and $w$ to build $M'$ (as above).
2. run $R$ on $\langle M' \rangle$
3. if $R$ accepts, reject.
   if $R$ rejects, accept.