A language is called **regular** iff some DFA recognizes it.

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Let $A$ and $B$ be languages

**Union:** $A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$

**Concatenation:** $A \cdot B = \{ xy \mid x \in A \text{ and } y \in B \}$

**Star:** $A^* = \{ x_1 x_2 x_3 \ldots x_k \mid k \geq 0 \text{ and each } x_i \in A \}$
\[ \Sigma' = \{ a \ldots z \} \]
\[ A = \{ \text{one, two} \} \]
\[ B = \{ \text{fish, bird} \} \]
\[ A \cup B = \{ \text{one, two, fish, bird} \} \]
\[ A \circ B = \{ \text{onefish, onebird, twofish, twobird} \} \]
\[ B \circ A = \{ \text{fishone, birdone, fishtwo, birdtwo} \} \]
\[ A^* = \{ \varepsilon, \text{one, two, oneone, onetwo, twoone, twotwo, \ldots} \} \]
DFA accepts all strings that begin w/ 1 or end w/ 1
\[ \Sigma = \{0, 1\} \]
\[ A_1 = \{ w \mid w \text{ begins with } 1 \} \]
\[ A_2 = \{ w \mid w \text{ ends with } 1 \} \]

\[ \begin{array}{c}
M_1 \\
1 \rightarrow 2 \\
2 \rightarrow 3 \\
3 \rightarrow \emptyset
\end{array} \]

\[ \begin{array}{c}
M_2 \\
A \rightarrow B
\end{array} \]

\[ A_1 \cup A_2 \]

\[ \begin{array}{c}
A_1 \leftrightarrow A_2 \\
A_2 \leftrightarrow A_3
\end{array} \]
The class of regular languages is closed under the union operation.

if \( A_1 \) and \( A_2 \) are regular
then \( A_1 \cup A_2 \) is regular

Proof:

Suppose \( A_1 \) and \( A_2 \) are regular. 
[we will show \( A_1 \cup A_2 \) is regular]

There is a DFA \( M_1 \) that accepts \( A_1 \) and \( M_2 \) that accepts \( A_2 \)

\[
M_1 = (Q_1, \Sigma, \delta_1, s_1, q_f^1)
\]
\[
M_2 = (Q_2, \Sigma, \delta_2, s_2, q_f^2)
\]

Construct \( M \) that recognizes \( A_1 \cup A_2 \)

\[
M = (Q, \Sigma, \delta, s, q_0, F)
\]

\[
Q = Q_1 \times Q_2 = \{ (r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2 \} \]

\[
\Sigma
\]

\[
\delta: \text{ For each } (r_1, r_2) \in Q, \text{ and } a \in \Sigma
\]

\[
\delta((r_1, r_2), a) = (\delta_1(r_1, a), \delta_2(r_2, a))
\]

\[
q_0 = (q_0^1, q_0^2)
\]

\[
F = \{ (r_1, r_2) \mid r_1 \in F_1 \text{ or } r_2 \in F_2 \}
\]

So \( M \) recognizes \( A_1 \cup A_2 \).
Therefore \( A_1 \cup A_2 \) is regular.