Cardinality

Two sets have same cardinality iff there is a bijection from one to the other.

$2\mathbb{Z} \subset \mathbb{Z}$

(even)

$H(n) = 2n \qquad H : \mathbb{Z} \to 2\mathbb{Z}$

one to one:

$$H(n_1) = H(n_2)$$
$$2n_1 = 2n_2$$
$$n_1 = n_2$$

onto: suppose $m \in 2\mathbb{Z}$

so $m = 2k = H(k)$

Countable Sets

$\mathbb{Z}^+$: counting numbers

Set S if $f : \mathbb{Z}^+ \to S$

then we can "count" the values in S

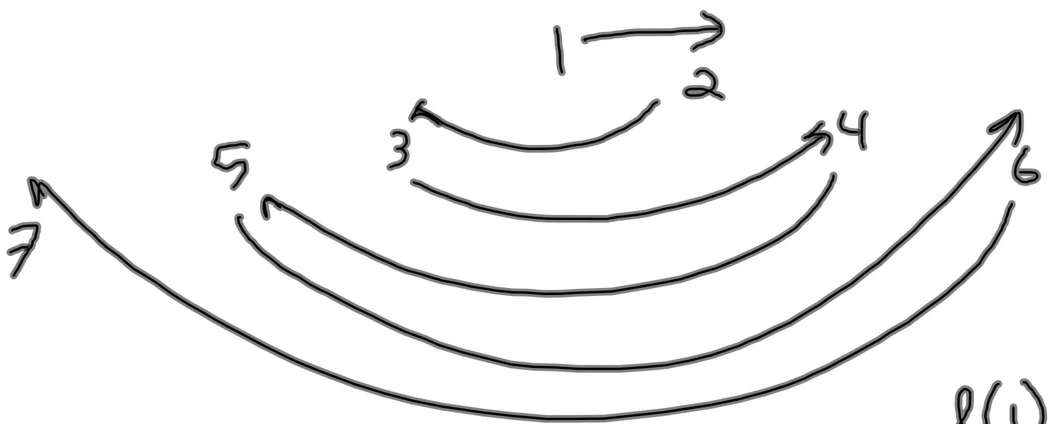
$$f(1) = s_1 \quad \text{first}$$
$$f(2) = s_2 \quad \text{second}$$
$$f(3) = s_3 \quad \text{third}$$
$$\vdots \qquad\qquad \vdots$$

A set is <u>countably infinite</u> iff it has the same cardinality $\mathbb{Z}^+$

A set is <u>countable</u> iff it is finite or countably infinite

$\mathbb{Z}$ is countable

$$\cdots -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \cdots$$



$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$f(1) = 0$

$f(2) = 1$

$f(3) = -1$

$f(4) = 2$

Rational Numbers? $\mathbb{Q}^+$

$$
\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \cancel{\frac{2}{2}} & \frac{2}{3} & \cancel{\frac{2}{4}} & \frac{2}{5} & \cdots \\
\frac{3}{1} & \frac{3}{2} & \cancel{\frac{3}{3}} & \frac{3}{4} & \frac{3}{5} & \cdots \\
\frac{4}{1} & \cancel{\frac{4}{2}} & \frac{4}{3} & \frac{4}{4} & \frac{4}{5} & \cdots \\
\vdots & & & & &
\end{array}
$$

reals between 0 and 1 = S
are uncountable

Proof by contradiction

suppose S is countable

so $\exists$ a bijection $f : \mathbb{Z}^+ \to S$

$a_{ij} \in \{0..9\}$

$f(1) = 0.\boxed{a_{11}}\, a_{12}\, a_{13}\, a_{14}\, a_{15} \dots a_{1n} \dots$

$f(2) = 0.a_{21}\, \boxed{a_{22}}\, a_{23}\, a_{24}\, a_{25} \dots a_{2n} \dots$

$f(3) = 0.a_{31}\, a_{32}\, \boxed{a_{33}}\, a_{34}\, a_{35} \dots a_{3n} \dots$

$\vdots$

$f(n) = 0.a_{n1}\, a_{n2}\, a_{n3}\, a_{n4}\, a_{n5} \dots \boxed{a_{nn}} \dots$

$\vdots$

construct $d = 0.d_1 d_2 d_3 \dots d_n \dots$

$$d_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{cases}$$

d differs from each item in at least 1 position

so $\nexists$ k s.t $f(k) = d$

which is a contradiction

e.g.

$f(1) = 0.\underline{2}3574\dots$

$f(2) = 0.4\underline{1}628\dots$

$f(3) = 0.74\underline{8}91\dots$

$\vdots$

$d = 0.121\dots$

P: set of all programs in a language
(strings of 1's and 0's)

1011, 011, 11, 01, 101

01, 11, 011, 101, 1011
by length then by 0,1 from
left to right