$f: X \to Y$

one-to-one

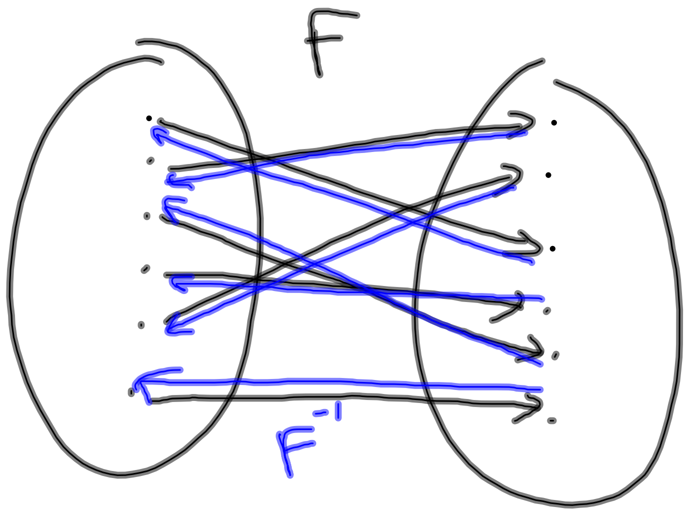
if $f(x_1) = f(x_2)$ then $x_1 = x_2$

onto

every y is mapped to by some x

One to one correspondence (bijection)

function is one to one and onto



if $F: X \to Y$ is a bijection

then there is a function $F^{-1}: Y \to X$

s.t. $F^{-1}(y) = x$ $\quad x \in X$

and x is unique

and $f(x) = y$

<u>Cardinality</u> - number of elements in a set.

Two sets A, B have the same cardinality iff there is a bijection from A to B.

$\mathbb{Z}$ are infinite

$2\mathbb{Z}$ (even numbers) are infinite

$2\mathbb{Z} \subset \mathbb{Z}$

Have the same cardinality

prove by constructing a bijection

$H: \mathbb{Z} \to 2\mathbb{Z}$

$H(n) = 2 * n$

show H is one-to-one

Suppose $H(n_1) = H(n_2)$

$\cancel{2} \cdot n_1 / 2 = \cancel{2} \cdot n_2 / 2$

$n_1 = n_2$

show H is onto

Suppose $m \in 2\mathbb{Z}$

so m is even

so $m = 2 \cdot k$, $k \in \mathbb{Z}$

so $H(k) = 2k = m$

$\mathbb{Z}^+$

find a bijection $f: \mathbb{Z}^+ \to S$

using f we can "count" the elements of S

S is <u>countably infinite</u> iff it has the same cardinality as $\mathbb{Z}^+$

T is <u>countable</u> iff it is countably infinite or finite

$\mathbb{Z}$ is countable

$$\ldots -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \ldots$$

$$\ldots \quad 7 \quad 5 \quad 3 \quad 1 \quad 2 \quad 4 \quad 6 \ldots$$

$$f: \mathbb{Z}^+ \to \mathbb{Z}$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$\mathbb{Q}^+$ rationals are countable

$$\begin{array}{cccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \frac{2}{1} & \cancel{\frac{2}{2}} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

$F(1) = \frac{1}{1}$

$F(2) = \frac{1}{2}$

$F(3) = \frac{2}{1}$

$\vdots$

allowed $\frac{2}{2}$

$F(1) = \frac{1}{1} = 1$

$F(5) = \frac{2}{2} = 1$

Real numbers between 0 and 1

they are uncountable

Diagonalization (Cantor)

Suppose they are countable ∞

so there is a bijection

$f : \mathbb{Z}^+ \rightarrow$ reals from $0..1$

so digits

$f(1) = 0.\boxed{a_{11}}\, a_{12}\, a_{13} \cdots a_{1n} \cdots$

$f(2) = 0.a_{21}\, \boxed{a_{22}}\, a_{23} \cdots a_{2n} \cdots$

$f(3) = 0.a_{31}\, a_{32}\, \boxed{a_{33}} \cdots a_{3n} \cdots$

$f(4) =$

$\vdots$

$f(n) = 0.a_{n1}\, a_{n2}\, a_{n3} \cdots \boxed{a_{nn}} \cdots$

$\vdots$

construct

$d = 0.d_1 d_2 d_3 \cdots d_n \cdots$

s.t. $d_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{cases}$

f does not map to d.

so f is not <u>onto</u>