Indirect Proof

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

$$\begin{array}{l} p \to q \\ p \wedge \sim q \\ \hline \therefore C \end{array}$$

proof by contradiction

(reductio ad absurdum)

1. Suppose the statement is false
   (Suppose negation is true)

2. Supposition leads to a contradiction

3. Conclude the orig. statement is true.

There is no greatest integer.

proof: (by contradiction)

Suppose there is a greatest integer, N.

① $\forall n \in \mathbb{Z},\ n \le N$

Let $M = N + 1$. so $M \in \mathbb{Z}$ and $M > N$.

which contradicts ①

The sum of any rational number
and any irrational number
is irrational.

$$\forall\, r \in \mathbb{Q}, s \notin \mathbb{Q}, \quad r+s \notin \mathbb{Q}$$

suppose (not)

$$\exists\, r \in \mathbb{Q}, \underline{s \notin \mathbb{Q}} \text{ s.t. } r+s \in \mathbb{Q}$$

$$r = \frac{a}{b} \qquad a,b \in \mathbb{Z} \quad b \neq 0$$

$$r+s = \frac{c}{d} \qquad c,d \in \mathbb{Z} \quad d \neq 0$$

$$\frac{a}{b} + s = \frac{c}{d}$$

$$s = \frac{c}{d} - \frac{a}{b}$$

$$= \frac{cb - ad}{db} \qquad \begin{matrix} d \neq 0 \\ b \neq 0 \end{matrix}$$

so $db \neq 0$

Let $p = cb - ad$
$q = db \quad (q \neq 0)$

so $s = \dfrac{p}{q}$ $\quad q \in 0$
$p \in \mathbb{Z}$
$q \in \mathbb{Z}$

so s is rational
but s was <u>not</u> rational
which is a contradiction.

Contraposition

1. Express statement as

$$\forall x \in D,\ P(x) \to Q(x)$$

2. Rewrite as contrapositive

$$\forall x \in D,\ \sim Q(x) \to \sim P(x)$$

3. Prove contrapositive as in a direct proof:

Suppose: $\sim Q(x)$

Show: $\sim P(x)$

For all $n \in \mathbb{Z}$, if n^2 is even then n is even.

contrapositive:

if n is odd then n^2 is odd.

suppose n is an odd integer.

$n = 2k+1 \quad k \in \mathbb{Z}$

$$n^2 = (2k+1)^2$$
$$= 4k^2 + 4k + 1$$
$$= 2(2k^2 + 2k) + 1$$

$\vdots$

for all $a, b, c \in \mathbb{Z}$,

$$a \nmid bc \rightarrow a \nmid b$$

$p \rightarrow q$

$p \wedge \neg q$

Proof by contradiction.

Suppose: $\exists\, a, b, c \in \mathbb{Z}$ s.t.

$a \nmid bc$ and $a \mid b$

since $a \mid b$, $b = a \cdot k$ $\quad k \in \mathbb{Z}$

so $bc = (ak)c$

$= (ck)a$

Let $l = ck$ $\quad l \in \mathbb{Z}$

so $bc = l \cdot a$

$\therefore a \mid bc$

which is a controdiction.

Therefore

$\forall a, b, c \in \mathbb{Z}$, $a \nmid bc \rightarrow a \nmid b$

for all $a, b, c \in \mathbb{Z}$,

$a \nmid bc \rightarrow a \nmid b$

proof by contraposition

$\forall a, b, c \in \mathbb{Z}, a \mid b \rightarrow a \mid bc$

suppose $a \mid b$ $a, b \in \mathbb{Z}$

show $a \mid bc$

since $a \mid b$ we have $b = a \cdot k$, $k \in \mathbb{Z}$

consider $b \cdot c = (ak) \cdot c$ subst

$= (c \cdot k) a$

Let $l = ck$

so $bc = l \cdot a$ $l \in \mathbb{Z}$

which by def. of $\mid$

$a \mid bc$.