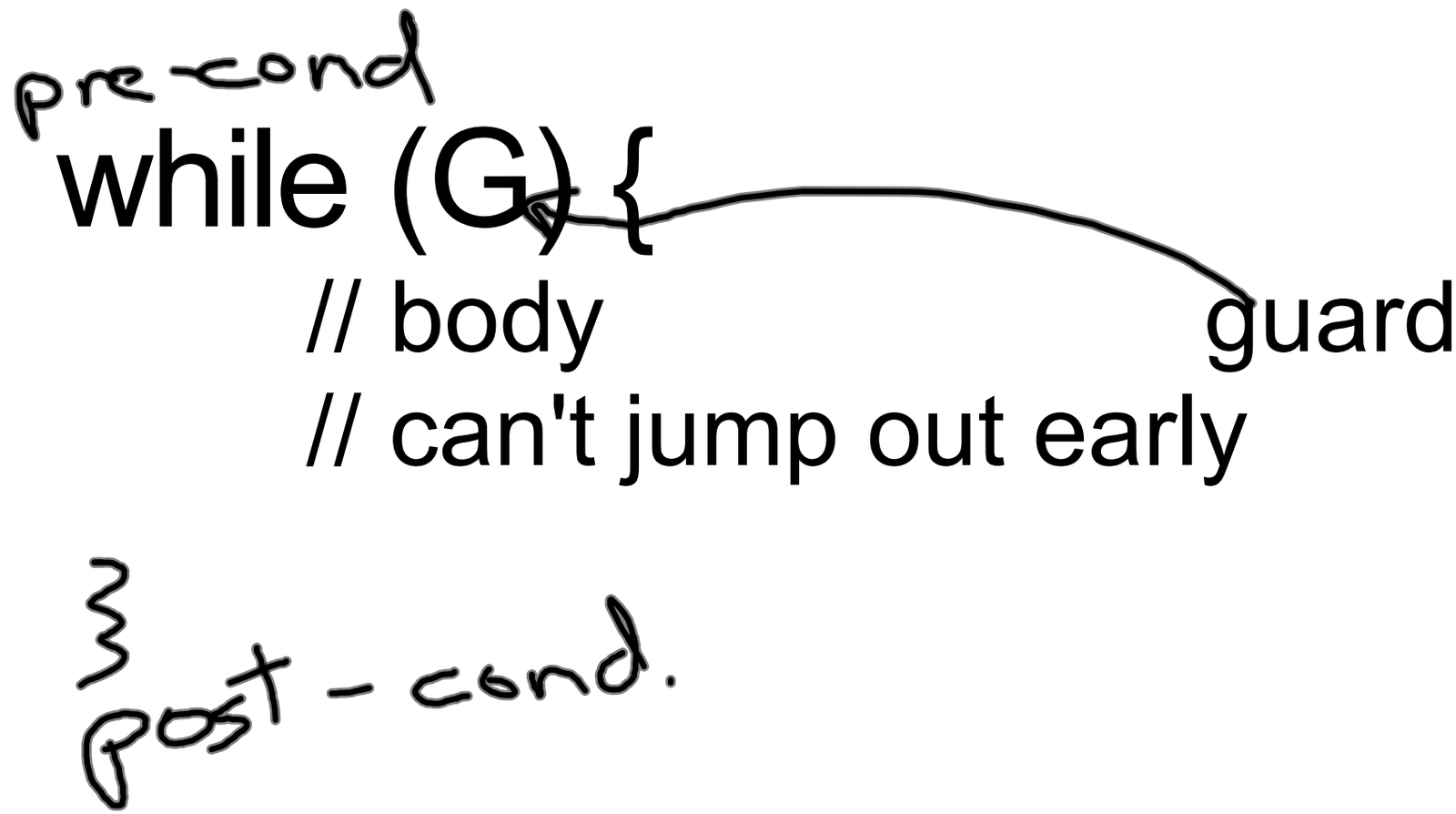
Correctness of loops (method of loop invariants)



Loop invariant

- Predicate over the set of integers

1. true before the first iteration $P(0)$

2. For each iteration
   if it is true before,
   then it is true after.

3 If the loop finishes after finite steps, then the truth of the invariant ensures the truth of the post-cond.

Given: a while loop
a guard G
an invariant $I(n)$

If the following are true,
then the loop is correct

I. Basis $I(0)$ is true

II. Induction $\forall k \geq 0, k \in \mathbb{Z}$
if $G \wedge I(k)$ then $I(k+1)$

III. Eventual falsity of G

IV. Correctness of the post condition

product $m \cdot x$

Pre-cond: m is a non-neg int, x is real, i = 0 and product = 0

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while ( i ≠ m)
    product := product + x
    i := i + 1
end while
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Postcondition: product = $m \cdot x$

Loop invariant: I(n): i = n and product = $n \cdot x$

I. Basis I(0)

show i=0 and product=0·x
true from precondition.

II. Induction: if I(k) ∧ G then I(k+1)

Suppose I(k) and G are true
for some k, prior to an
iteration of the loop
Show I(k+1) is true after that
iteration.

I(k) is true so i=k
and product = k·x

G is true so i≠m

Since G is true, statement 1
is executed.

$product_{new} = product_{old} + x$

and $product_{old} = k \cdot x$

so $product_{new} = k \cdot x + x$ subst. ind. hyp

$= (k+1) \cdot x$

Also stat. 2 is executed

so $i_{new} = i_{old} + 1$

and $i_{old} = k$

so $i_{new} = k+1$

So I(k+1) is true.

III Eventual falsity of the Guard.

G: $i \neq m$

m is a non-neg. int

from I and II

for all $n \geq 0$ if the loop is iterated n times then $i = n$ and product $= n \cdot x$

So after m iterations $i = m$.

So G is false

IV Correctness of the post condition

post-cond: product $= m \cdot x$ after execution of the loop

G is false after m iterations (from III)

and I(m) was true

so product $= mx$

QED

Pre-cond	$i=0$	product$=0$
$I(0)$	0	0
$I(1)$	1	x
$I(2)$	2	$2 \cdot x$
$\vdots$		
$I(m)$	m	$m \cdot x$
post-cond		$m \cdot x$